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On the construction of general cubature formula by flat extensions

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ABSTRACT

We describe a new method to compute general cubature formulae. The problem is initially transformed into the computation of truncated Hankel operators with flat extensions. We then analyze the algebraic properties associated to flat extensions and show how to recover the cubature points and weights from the truncated Hankel operator. We next present an algorithm to test the flat extension property and to additionally compute the decomposition. To generate cubature formulae with a minimal number of points, we propose a new relaxation hierarchy of convex optimization problems minimizing the nuclear norm of the Hankel operators. For a suitably high order of convex relaxation, the minimizer of the optimization problem corresponds to a cubature formula. Furthermore cubature formulae with a minimal number of points are associated to faces of the convex sets. We illustrate our method on some examples, and for each we obtain a new minimal cubature formula.

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1. Cubature formula

1.1. Statement of the problem

Consider the integral for a continuous function f,

$$I[f] = \int_{\Omega} w(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}$$

where $\Omega \subset \mathbb{R}^n$ and w is a positive function on Ω.

We are looking for a cubature formula which has the form

$$\langle \sigma | f \rangle = \sum_{j=1}^{r} w_j f(\zeta_j) \tag{1}$$

where the points $\zeta_j \in \mathbb{C}^n$ and the weights $w_j \in \mathbb{R}$ are independent of the function f. They are chosen so that

$$\langle \sigma | f \rangle = I[f], \forall f \in V,$$

where V is a finite dimensional vector space of functions. Usually, the vector space V is the vector space of polynomials of degree $\leq d$, because a well-behaved function f can be approximated by a polynomial, so that $Q[f]$ approximates the integral $I[f]$.

Given a cubature formula (1) for I, its algebraic degree is the largest degree d for which $I[f] = \langle \sigma | f \rangle$ for all f of degree $\leq d$.

1.2. Related works

Prior approaches to the solution of cubature problem can be grouped into roughly two classes. One, where the goal is to estimate the fewest weighted, aka cubature points possible for satisfying a prescribed cubature rule of fixed degree [9,24,26,29,30,33]. The other class focusses on the determination and construction of cubature rules which would yield the fewest cubature points possible [7,34,38–41,44,45]. In [34], for example, Radon introduced a fundamental technique for constructing minimal cubature rules where the cubature points are common zeros of multivariate orthogonal polynomials. This fundamental technique has since been extended by many, including e.g. [33,41,45] where notably, the paper [45] uses multivariate ideal theory, while [33] uses operator dilation theory. In this paper, we propose another approach to the second class of cubature solutions, namely, constructing a suitable finite dimensional Hankel matrix and extracting the cubature points using sub-operators of the Hankel matrix [18]. This approach is related to [21–23], which in turn are based on the methods of multivariate truncated moment matrices, their positivity and extension properties [11–13].