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Sparse block factorization of saddle point matrices

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ABSTRACT

The factorization method presented in this paper takes advantage of the special structures and properties of saddle point matrices. A variant of Gaussian elimination equivalent to the Cholesky's factorization is suggested and implemented for factorizing the saddle point matrices block-wise with small blocks of orders 1 and 2. The Gaussian elimination applied to these small blocks on block level also induces a block 3×3 structured factorization of which the blocks have special properties. We compare the new block factorization with the Schilders' factorization in terms of sparsity and computational complexity. The factorization can be used as a direct method, and also anticipate for preconditioning techniques.

1. Introduction

Indefinite matrices with special forms which occur in many scientific and engineering problems can be exploited efficiently by taking advantage of the structures and properties

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of their blocks. We consider symmetric indefinite linear systems of the form (see [2] for the notation)

$$\mathring{\boldsymbol{\mathcal{A}}}\,\mathring{\boldsymbol{u}} = \mathring{\boldsymbol{b}} \text{ with } \mathring{\boldsymbol{\mathcal{A}}} = \begin{bmatrix} \mathring{\boldsymbol{A}} & \mathring{\boldsymbol{B}}^T \\ \mathring{\boldsymbol{B}} & \boldsymbol{0} \end{bmatrix},\ \mathring{\boldsymbol{u}} = \begin{bmatrix} \mathring{\boldsymbol{x}} \\ \mathring{\boldsymbol{y}} \end{bmatrix},\ \mathring{\boldsymbol{b}} = \begin{bmatrix} \mathring{\boldsymbol{f}} \\ \mathring{\boldsymbol{g}} \end{bmatrix} \tag{1}$$

where $\mathring{\boldsymbol{A}} \in \mathbb{R}^{n\times n}$ is symmetric positive definite; $\mathring{\boldsymbol{B}} \in \mathbb{R}^{m\times n}$ has full rank and $m < n$; $\mathring{\boldsymbol{x}}, \mathring{\boldsymbol{f}} \in \mathbb{R}^n$; and $\mathring{\boldsymbol{y}}, \mathring{\boldsymbol{g}} \in \mathbb{R}^m$. In applications, the coefficient matrix $\mathring{\boldsymbol{\mathcal{A}}}$ is usually sparse and large, which can easily turn out to be a million by million. Systems of the form (1) are known as saddle point problems, which result from discretization of PDEs or coupled PDEs such as the Stokes problem and generally in the context of mixed finite element methods. Saddle point systems also arise in electronic circuit simulations [28, 32], Maxwell's equations [22], economic models and constrained optimization problems [9,14,15,17,30]. For example consider the equality-constrained quadratic programming problem:

$$\min_{\mathring{\boldsymbol{x}}} p(\mathring{\boldsymbol{x}}) = \frac{1}{2}\mathring{\boldsymbol{x}}^T \mathring{\boldsymbol{A}}\,\mathring{\boldsymbol{x}} - \mathring{\boldsymbol{x}}^T \mathring{\boldsymbol{f}} \quad \text{subject to} \quad \mathring{\boldsymbol{B}}\,\mathring{\boldsymbol{x}} = \mathring{\boldsymbol{g}}. \tag{2}$$

The Karush–Kuhn–Tucker (KKT) conditions [14,36] for the solution to (2) give rise to the system (1), where the components of $\mathring{\boldsymbol{y}}$ are the associated Lagrange multipliers. Thus the coefficient matrix $\mathring{\boldsymbol{\mathcal{A}}}$ is also known as KKT matrix and it is nonsingular if (i) $\mathring{\boldsymbol{B}}$ has full row rank and (ii) the reduced Hessian matrix $\mathring{\boldsymbol{Z}}^T \mathring{\boldsymbol{A}}\,\mathring{\boldsymbol{Z}}$ is positive definite, where $\mathring{\boldsymbol{Z}} \in \mathbb{R}^{n\times(n-m)}$ is the matrix whose columns span the $\ker(\mathring{\boldsymbol{B}})$ [26, p. 443].

Numerous solution methods for the saddle point systems of the form (1) can be found in the literature and many of them have focused on preconditioning techniques for Krylov subspace iterative solvers [2,6,11,17,20–22,25,29]. One can find a wide range of iterative methods in [1], a detailed survey done by Benzi, Golub and Liesen including preconditioning techniques. As a direct method (as opposed to iterative solvers), various techniques based on symmetric indefinite factorization $\boldsymbol{P}^T \mathring{\boldsymbol{\mathcal{A}}}\boldsymbol{P} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$ can be found in [8,13,19,31,32,35], where $\boldsymbol{P}$ is a permutation matrix, $\boldsymbol{L}$ is unit lower triangular matrix, $\boldsymbol{D}$ is block-diagonal matrix with blocks of order 1 or 2. The permutation matrix $\boldsymbol{P}$ is introduced for (i) *pivoting dynamically* and (ii) *reducing the fill-in* in $\boldsymbol{L}$ if $\mathring{\boldsymbol{\mathcal{A}}}$ is sparse. The block diagonal pivoting strategies are mainly due to Bunch and Kaufman [3], Bunch and Parlett [4] and Bunch, Kaufman and Parlett (BKP) [5].

In this paper, we propose a different transformation $\boldsymbol{\tau}^T \mathring{\boldsymbol{\mathcal{A}}}\boldsymbol{\tau} = \boldsymbol{\mathcal{A}}$, followed by a block Gaussian elimination factorization $\boldsymbol{P_\pi}^T \boldsymbol{\mathcal{A}}\boldsymbol{P_\pi} = \boldsymbol{L_b}\,\boldsymbol{D_b}^{-1}\,\boldsymbol{L_b}^T$, where:

(i) $\boldsymbol{L_b}$ is block lower triangular with blocks of orders 1 and 2, and $\boldsymbol{D_b} = \operatorname{diag}(\boldsymbol{L_b})$ is the block diagonal part of $\boldsymbol{L_b}$ with blocks of orders 1 and 2.

[2] The original system (1) undergoes a transformation, so we use the symbol '∘' in its notation in order to represent the transformed system more conveniently without the symbol '∘'.