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Adjusted least squares fitting of algebraic hypersurfaces

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A B S T R A C T

We consider the problem of fitting a set of points in Euclidean space by an algebraic hypersurface. We assume that points on a true hypersurface, described by a polynomial equation, are corrupted by zero mean independent Gaussian noise, and we estimate the coefficients of the true polynomial equation. The adjusted least squares estimator accounts for the bias present in the ordinary least squares estimator. The adjusted least squares estimator is based on constructing a quasi-Hankel matrix, which is a bias-corrected matrix of moments. For the case of unknown noise variance, the estimator is defined as a solution of a polynomial eigenvalue problem. In this paper, we present new results on invariance properties of the adjusted least squares estimator and an improved algorithm for computing the estimator for an arbitrary set of monomials in the polynomial equation.

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1. Introduction

An algebraic hypersurface is the set of points $d \in \mathbb{R}^{\mathtt{q}}$ that are the solutions of an *implicit polynomial equation*

$$R_\theta(d) = 0. \tag{1}$$

In (1), $R_\theta(d)$ is a multivariate polynomial with coefficients $\theta = [\,\theta_1 \quad \cdots \quad \theta_m\,]^\top$

$$R_\theta(d) := \theta^\top \phi(d) = \sum_{j=1}^{m} \theta_j \phi_j(d), \tag{2}$$

where $\phi(d)$ is the vector of linearly independent basis polynomials

$$\phi(d) = [\,\phi_1(d) \quad \cdots \quad \phi_m(d)\,]^\top, \quad d \in \mathbb{R}^{\mathtt{q}}. \tag{3}$$

The algebraic hypersurface fitting problem is to fit a given set of points

$$\mathscr{D} = \{d^{(1)}, \ldots, d^{(N)}\} \subset \mathbb{R}^{\mathtt{q}},$$

in the best way by an algebraic hypersurface of the form (1), where the vector of basis monomials is given and fixed. The notion of "best" is determined by a chosen goodness-of-fit measure.

Fitting two-dimensional data by conic sections ($\mathtt{q} = 2$) is the most common case of algebraic hypersurface fitting, with numerous applications in robotics, medical imaging, archaeology, etc., see [1] for an overview. Fitting algebraic hypersurfaces of higher degrees and dimensions is needed in computer graphics [2], computer vision [3], and symbolic–numeric computations [4], [5, §5]. The problem also appears in advanced methods of multivariate data analysis such as subspace clustering [6] and non-linear system identification [7], see [8] for an overview. Algebraic hypersurface fitting received considerable attention in linear algebra community starting from [9]. Recently, it has been shown to be an instance of nonlinearly structured low-rank approximation [10, Ch. 6].

The most widespread fits are *geometric* and *algebraic* fit (see, for example, [9] and [11]). The geometric fit minimizes total distance from $\mathscr{D}$ to a hypersurface defined by (1). Although this is a natural goodness-of-fit measure, the resulting nonlinear optimization problem is difficult.

A computationally cheap alternative to geometric fit is the algebraic fit, which minimizes the sum of squared residuals of the implicit equation (1)

$$Q_{\mathrm{ols}}(\theta, \mathscr{D}) := \sum_{k=1}^{N} \left(R_\theta(d^{(k)}) \right)^2. \tag{4}$$