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Computing the exponential of large block-triangular block-Toeplitz matrices encountered in fluid queues

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ABSTRACT

The Erlangian approximation of Markovian fluid queues leads to the problem of computing the matrix exponential of a subgenerator having a block-triangular, block-Toeplitz structure. To this end, we propose some algorithms which exploit the Toeplitz structure and the properties of generators. Such algorithms allow us to compute the exponential of very large matrices, which would otherwise be untreatable with standard methods. We also prove interesting decay properties of the exponential of a generator having a block-triangular, block-Toeplitz structure.

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1. Introduction

The problem we consider here is to compute the exponential of an upper block-triangular, block-Toeplitz matrix, that is, a matrix of the kind

$$\mathcal{T}(U) = \begin{bmatrix} U_0 & U_1 & \dots & U_{n-1} \\ & U_0 & \ddots & \vdots \\ & & \ddots & U_1 \\ 0 & & & U_0 \end{bmatrix}, \tag{1}$$

where U_i, $i = 0, \dots, n-1$, are $m \times m$ matrices. Our interest stems from the analysis in Dendievel and Latouche [1] of the Erlangization method for Markovian fluid models, but the story goes further back in time.

1.1. Origin of the problem

The Erlangian approximation method was introduced in Asmussen et al. [2] in the context of risk processes; it was picked up in Stanford et al. [3] where a connection is established with fluid queues. Other relevant references are Stanford et al. [4] where the focus is on modelling the spread of forest fires, and Ramaswami et al. [5] where some basic algorithms are developed.

Markovian fluid models are two-dimensional processes $\{(X(t), \varphi(t)) : t \in \mathbb{R}^+\}$ where $\{\varphi(t)\}$ is a Markov process with infinitesimal generator A on the state space $\{1, \dots, m\}$; to each state i is associated a rate of growth $c_i \in \mathbb{R}$ and $X(t)$ is controlled by $\varphi(t)$ through the equation

$$X(t) = X(0) + \int_0^t c_{\varphi(s)}\, \mathrm{d}s, \qquad \text{for } t \geq 0.$$

Performance measures of interest include the distributions of $X(t)$ and of various first passage times. Usually, $\varphi(t)$ is called the phase of the process at time t and $X(t)$ its level, and the phase space $\{1, \dots, m\}$ is partitioned into three subsets $\mathcal{S}_+$, $\mathcal{S}_-$ and $\mathcal{S}_0$ such that $c_i > 0$, $c_i < 0$ or $c_i = 0$ if i is in $\mathcal{S}_+$, $\mathcal{S}_-$ or $\mathcal{S}_0$, respectively. To simplify our presentation without missing any important feature, we assume below that $\mathcal{S}_0$ is empty.

The first return probabilities of $X(t)$ to its initial level $X(0)$ play a central role in the analysis of fluid queues. It is customary to define two matrices Ψ and $\hat{\Psi}$ of the first return probabilities:

$$\Psi_{ij} = \Pr[\tau < \infty, \varphi(\tau) = j | X(0) = 0, \varphi(0) = i], \qquad i \in \mathcal{S}_+,\ j \in \mathcal{S}_-,$$

and