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Simultaneous similarity classes of commuting matrices over a finite field

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ABSTRACT

This paper concerns the enumeration of isomorphism classes of modules of a polynomial algebra in several variables over a finite field. This is the same as the classification of commuting tuples of matrices over a finite field up to simultaneous similarity. Let $c_{n,k}(q)$ denote the number of isomorphism classes of n-dimensional $\mathbb{F}_q[x_1, \ldots, x_k]$-modules. The generating function $\sum_k c_{n,k}(q)t^k$ is a rational function. We compute this function for $n \leq 4$. We find that its coefficients are polynomial functions in q with non-negative integer coefficients.

1. Introduction

1.1. Background

Let $\mathbb{F}_q$ be a finite field of order q. Let $\mathcal{A}$ be a finite dimensional algebra (with unity) over $\mathbb{F}_q$. Let $\mathcal{A}^*$ denote the group of units in $\mathcal{A}$. Let $k \geq 1$ be a positive integer, and $\mathcal{A}^{(k)}$ be the set of k-tuples of elements of $\mathcal{A}$, whose entries commute with each other, i.e.,

$$\mathcal{A}^{(k)} = \{(a_1, \ldots, a_k) \in \mathcal{A}^k \mid a_i a_j = a_j a_i \text{ for } i \neq j\}.$$

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Denote by $c_{\mathcal{A},k}$, the number of orbits in $\mathcal{A}^{(k)}$ for the action of simultaneous conjugation of $\mathcal{A}^*$ on it, which is defined below:

Definition 1.1. For $(a_1, \ldots, a_k) \in \mathcal{A}^{(k)}$ and $g \in \mathcal{A}^*$:

$$g.(a_1, \ldots, a_k) = (ga_1g^{-1}, \ldots, ga_kg^{-1}).$$

This action is called **simultaneous conjugation**, and the orbits for this action are called **simultaneous similarity classes**.

For an element, $a \in \mathcal{A}$, let $Z_{\mathcal{A}}(a)$ denote the centralizer algebra of a, and $Z_{\mathcal{A}^*}(a)$ denote the group of units of $Z_{\mathcal{A}}(a)$. Let $h_{\mathcal{A}}(t)$ be the generating function of $c_{\mathcal{A},k}$ in k:

$$h_{\mathcal{A}}(t) = 1 + \sum_{k=1}^{\infty} c_{\mathcal{A},k} t^k.$$

Then we have the theorem:

Theorem 1.2. *For any finite dimensional algebra $\mathcal{A}$ over $\mathbb{F}_q$, $h_{\mathcal{A}}(t)$ is a rational function.*

Proof. Given $a_1 \in \mathcal{A}$, consider the k-tuple, $(a_1, a_2, \ldots, a_k) \in \mathcal{A}^{(k)}$. That means, $(a_2, \ldots, a_k) \in Z_{\mathcal{A}}(a_1)^{(k-1)}$. Thus, the map:

$$(a_1, a_2, \ldots, a_k) \mapsto (a_2, \ldots, a_k),$$

induces a bijection from the set of $\mathcal{A}^*$-orbits in $\mathcal{A}^{(k)}$, which contain an element whose 1st coordinate is a_1, onto the set of orbits in $Z_{\mathcal{A}}(a_1)^{(k-1)}$ for the action of simultaneous conjugation by $Z_{\mathcal{A}^*}(a_1)$ on it. Thus, we get:

$$c_{\mathcal{A},k} = \sum_{Z \subseteq \mathcal{A}} s_Z c_{Z,k-1},$$

where Z runs over subalgebras of $\mathcal{A}$, s_Z is the number of similarity classes in $\mathcal{A}$ whose centralizer algebra is isomorphic to Z, and $c_{Z,k-1}$ is the number of orbits under the action of Z^* (the group of units of Z) on $Z^{(k-1)}$ by simultaneous conjugation. Hence, we have

$$\begin{aligned} h_{\mathcal{A}}(t) &= 1 + \sum_{k=1}^{\infty} c_{\mathcal{A},k} t^k \\ &= 1 + \sum_{k=1}^{\infty} \left(\sum_{Z \subseteq \mathcal{A}} s_Z c_{Z,k-1} \right) t^k \end{aligned}$$