Linear Algebra and its Applications 501 (2016) 123–161

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Potentials of random walks on trees

Claude Dellacherie [a], Servet Martinez [b], Jaime San Martin [b,*]

[a] *Laboratoire Raphaël Salem, UMR 6085, Université de Rouen, Site du Madrillet, 76801 Saint Étienne du Rouvray Cedex, France*
[b] *CMM-DIM, Universidad de Chile, UMI-CNRS 2807, Casilla 170-3 Correo 3, Santiago, Chile*

A R T I C L E   I N F O

Article history:
Received 6 October 2015
Accepted 15 March 2016
Available online 25 March 2016
Submitted by V. Mehrmann

MSC:
15A48
15A51
60J45

Keywords:
M-matrix
Potential matrix
Tree matrices
Random walk on trees

A B S T R A C T

In this article we characterize inverse M-matrices and potentials whose inverses are supported on trees. In the symmetric case we show they are a Hadamard product of tree ultrametric matrices, generalizing a result by Gantmacher and Krein [12] done for inverse tridiagonal matrices. We also provide an algorithm that recognizes when a positive matrix W has an inverse M-matrix supported on a tree. This algorithm has quadratic complexity. We also provide a formula to compute W^{-1}, which can be implemented with a linear complexity. Finally, we also study some stability properties for Hadamard products and powers.

1. Introduction and basic definitions

Markov chains are used to model a variety of phenomena and one is accustomed to estimate its transition probability P in order to simulate and understand the underline probabilistic structure. Nevertheless, there are some situations where a direct measurement of P is not available. For example, this happens in electrical networks, where P is

* Corresponding author.
E-mail addresses: Claude.Dellacherie@univ-rouen.fr (C. Dellacherie), smartine@dim.uchile.cl (S. Martinez), jsanmart@dim.uchile.cl (J. San Martin).

http://dx.doi.org/10.1016/j.laa.2016.03.025
0024-3795/

related to the resistances of the network (see for example [10] section 2.7). Instead, we can measure the potential W of the Markov chain, which is the mean expected number of visits per site (assuming the chain is transient). These two matrices are related by $W = (\mathbb{I} - P)^{-1}$, so in principle one can model W instead of P. The main drawback of this approach is that structural restrictions for potentials are difficult to state (this is part of the inverse M-matrix problem).

In this article, we show how to handle this problem under the extra hypothesis that the incidence graph of W^{-1} is a tree. In the nomenclature of Klein [15], W^{-1} is a **treediagonal**. This is the case of a linear tree like Birth and Death chains and some networks like the electric power distribution system. In Theorem 2.5, we show that to reconstruct the chain it is enough to measure the potential at edges and nodes of the tree. The unique restrictions on those numbers are given by the 2×2 determinants associated with every edge. On the other hand, Theorem 2.2 (see also Corollary 2.3) provides an explicit formula to compute P from W. The complexity of this formula is linear in the number of nodes in the tree. More explicitly, if W is an $n \times n$ matrix, then the algorithm obtained from this formula, uses at most $11n$ operations (products, divisions and sums) to compute W^{-1}. Notice here that while W^{-1} is an sparse matrix, W itself is a full matrix (under irreducibility $W > 0$).

For the sake of completeness, we recall that a matrix Q is an **M-matrix** if it is a Z-matrix, that is, the off diagonal elements are nonpositive, Q is nonsingular and the entries of its inverse Q^{-1} are nonnegative. We refer to [13] section 2.5 for a set of equivalent conditions that characterize M-matrices. It is worth mentioning that the diagonal entries of an M-matrix are positive. On the other hand, a relevant sufficient condition for a Z-matrix Q to be an M-matrix, is that Q is nonsingular and **row diagonally dominant**, that is, for all i the row sum $\sum_j Q_{ij} \geq 0$.

In Theorem 2.7 we characterize, in an algorithmic way, those positive matrices whose inverses are M-matrices supported on a tree. The associated algorithm is developed in Appendix B, which provides the tree associated with W^{-1} with a complexity bounded by $\frac{37}{2}n^2$.

As a complement, Theorem 2.1 provides a description of a potential associated with a Markov chain supported on a tree. This is done in terms of ultrametric matrices. A sufficient condition is that $U = \mathrm{diag}(1./W_{\bullet r})\, W \mathrm{diag}(1./W_{r\bullet})$ is an ultrametric matrix. On the other hand, in the symmetric case, a necessary and sufficient condition is that W is the Hadamard product of tree ultrametric matrices, plus condition (2.2). As tree ultrametric matrices are simple to construct, this result provides a simple way to describe general Markov chains on trees.

In [9], we have proved that every potential of a random walk on $\{1, \cdots, n\}$ with nearest neighbor transitions, is the product of a positive diagonal matrix with a matrix which is the Hadamard product of two ultrametric matrices. This is equivalent to representing the inverse of a tridiagonal and row diagonally dominant M-matrix as such product. This was done in the symmetric case in [12]. In our setting, we shall see that we require one ultrametric matrix per extremal point of the set (on the tree) where the chain is losing