Linear Algebra and its Applications 501 (2016) 254–262

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Operators on triangular algebras

M. Sumanth Datt [a], Uma N. Iyer [b,*], G. Koteswara Rao [a]

[a] *University of Hyderabad, Prof. C.R Rao Road, P.O. Central University, Hyderabad 500 046, Telangana State, India*
[b] *Department of Mathematics and Computer Science, CP 315, 2155 University Avenue, Bronx, NY 10453, USA*

ARTICLE INFO

Article history:
Received 7 February 2016
Accepted 21 March 2016
Available online 29 March 2016
Submitted by P. Semrl

MSC:
16S32
13N10

Keywords:
Triangular algebra
Differential operators
Ideals

ABSTRACT

We study the algebra of differential operators on the triangular algebras and the upper triangular algebras. We further identify all the ideals of the algebra of differential operators on the upper triangular algebras.

1. Introduction

Let $\Bbbk$ be a field, $\bigotimes := \bigotimes_{\Bbbk}$, and all $\Bbbk$-algebras under consideration are associative and unital.

Let A be a $\Bbbk$-algebra. In [10], the definition of the algebra of differential operators on any $\Bbbk$-algebra is given. The second named author and T. McCune have identified these algebras of differential operators for several $\Bbbk$-algebras; see [4–7].

* Corresponding author.
E-mail addresses: sumanth.datt@gmail.com (M.S. Datt), uma.iyer@bcc.cuny.edu (U.N. Iyer), gullakoti@gmail.com (G.K. Rao).

http://dx.doi.org/10.1016/j.laa.2016.03.038

In this paper we consider the case when A is the triangular matrix algebra, or simply triangular algebras, constructed by using two $\Bbbk$ algebras R, S and an (R, S)-bimodule B, denoted by $T(R, B, S)$ (section 3) and when $A = U(R)$ is the algebra of upper triangular matrices over a $\Bbbk$-algebra R.

Various maps on triangular algebras, have been studied; for example, see [2] and [11] and references therein. Derivations, Jordan derivations, and automorphisms of $U(R)$ have been studied; see [1,3,8,9], and references therein. For instance, it has been established in these works that every $\Bbbk$-linear derivation on $U(R)$ is the sum of an inner derivation and an extension of a $\Bbbk$-linear derivation on R. Likewise, every $\Bbbk$-automorphism of $U(R)$ is the composition of an inner automorphism with an extension of a $\Bbbk$-automorphism of R. In this article we prove that every differential operator on $U(R)$ is given by an inner differential operator and an extension of a differential operator on R. This follows from the fact that $U(R) = U(\Bbbk) \otimes R$ implies that the algebra of differential operators on $U(R)$, $D_\Bbbk(U(R))$, is isomorphic to $D_\Bbbk(U(\Bbbk)) \otimes D_\Bbbk(R)$ (by Theorem 3.1.1 of [5]). We therefore focus this paper on the study of $D_\Bbbk(U(\Bbbk))$ and its ideals. We identify all the left, right, and two-sided ideals of $D_\Bbbk(U(\Bbbk))$ (Theorem 5.2). We also identify all the maximal left ideals of $D_\Bbbk(U(\Bbbk))$ (Theorem 5.1).

The paper is organized as follows. In section 2, we present the preliminaries required for the rest of the paper. In section 3 we identify the algebra of differential operators on the triangular algebra. We also describe the relations among the generating operators for the algebra. In section 4 we describe the algebra of differential operators on $U(\Bbbk)$.

2. Preliminaries

We recall the definition of the *algebra of differential operators* on A, denoted by $D_\Bbbk(A)$ from [10]. Let $\mathrm{Hom}_\Bbbk(A, A)$ denote the $\Bbbk$-algebra of $\Bbbk$-linear homomorphisms on A.

For each $r \in A$ we let $\lambda_r, \rho_r \in \mathrm{Hom}_\Bbbk(A, A)$ be given by $\lambda_r(s) = rs$ and $\rho_r(s) = sr$ for $s \in A$. For a homomorphism $\varphi \in \mathrm{Hom}_\Bbbk(A, A)$ and $r \in A$ when we write $[\varphi, r]$, we mean $[\varphi, \lambda_r]$.

The algebra $\mathrm{Hom}_\Bbbk(A, A)$ is an A-bimodule with $r \cdot \varphi \cdot s := \lambda_r \varphi \lambda_s$ for $r, s \in A$ and $\varphi \in \mathrm{Hom}_\Bbbk(A, A)$.

We now define the A-bimodule $D^m_\Bbbk(A)$ of differential operators on A of order $\leq m$. For $m \lneq 0$, we let $D^m_\Bbbk(A) = 0$. For $m \geq 0$, define $D^m_\Bbbk(A)$ to be the A-bimodule generated by

$$Z_m = \{\varphi \in Hom_\Bbbk(A, A) | \varphi r - r\varphi \in D^{m-1}_\Bbbk(A), \text{ for all } r \in A\}$$

We see that $\varphi \in Z_0$ implies, $\varphi(r) = (\varphi r)(1) = (r\varphi)(1) = r\varphi(1)$. Thus, $\varphi = \rho_{\varphi(1)}$, the homomorphism given by right multiplication by $\varphi(1)$. Thus, the A-bimodule generated by Z_0 is generated by the set of all homomorphisms λ_r, ρ_r, for $r \in A$, where λ_r is the homomorphism given by left multiplication by r. We therefore call elements of $D^0_\Bbbk(A)$ the *inner differential operators* on A.