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The Hadamard product for the weighted Karcher means

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ABSTRACT

The weighted Karcher mean of positive definite matrices is defined as the unique minimizer of the weighted sum of squares of the Riemannian distances to each of given points. Using the well-known connection between the tensor product and the Hadamard product, we show that the Hadamard product of weighted Karcher means for permuted tuples with fixed weight is bounded by the Hadamard product of given positive definite matrices. It generalizes the results for the case of two-variable geometric means and Sagae–Tanabe inductive means.

1. Introduction

In 1975 Pusz and Woronowicz [12] have introduced the notion of the geometric mean of two positive definite matrices A and B:

$$A\#B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}.$$

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Since then, T. Ando [1] has developed the robust definition of the geometric mean $A\#B$ with a variety of properties. One of the interesting properties of the geometric mean is the interaction with the Hadamard product (or called the Schur product). In other words, for two positive definite matrices A and B

$$(A\#B) \circ (A\#B) \leq A \circ B, \tag{1.1}$$

where $A \circ B = [a_{ij}b_{ij}]$ for $A = [a_{ij}]$ and $B = [b_{ij}]$. Here, the relation $\leq$ is the Löewner order defined as

$$A \leq B \text{ if and only if } B - A \text{ is positive semidefinite.}$$

If A and B commute, then the inequality (1.1) reduces to

$$(AB)^{1/2} \circ (AB)^{1/2} \leq A \circ B.$$

Using a different method, he also succeeded in generalizing this inequality to the case of several commuting positive definite matrices:

$$\prod_{1}^{m} \circ \left(\prod_{i=1}^{m} A_i \right)^{1/m} \leq \prod_{i=1}^{m} \circ A_i = A_1 \circ \cdots \circ A_m \tag{1.2}$$

for commuting positive definite matrices $A_1, \ldots, A_m$.

On the other hand, the inequality (1.2) has not been developed for the multivariable geometric means of several non-commuting positive definite matrices until M. Sagae and K. Tanabe [13] in 1994 successfully suggested the multivariable geometric mean. For given n-tuple of positive definite matrices $\mathbb{A} = (A_1, A_2, \ldots, A_n)$ and a positive probability vector $\omega = (w_1, \ldots, w_n)$, the definition $G_\omega(\mathbb{A})$ constructed by M. Sagae and K. Tanabe is as follows:

$$G_\omega(\mathbb{A}) := A_n \#_{\omega^{n-1}} (A_{n-1} \#_{\omega^{n-2}} \cdots \#_{\omega^2} (A_2 \#_{\omega^1} A_1)),$$

where for $1 \leq k \leq n-1$

$$\omega^k = 1 - w_{k+1} \left(\sum_{j=1}^{k+1} w_j \right)^{-1}.$$

We call $G_\omega(\mathbb{A})$ the *Sagae–Tanabe geometric mean*. B. Feng and A. Tonge in [4] have extended the inequality (1.2) to the Sagae–Tanabe mean $G_\omega(\mathbb{A})$ of several non-commuting positive definite matrices.