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Product of two positive contractions

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Several characterizations are given for a square matrix that can be written as the product of two positive (semidefinite) contractions. Based on one of these characterizations, and the theory of alternating projections, a Matlab program is written to check the condition and construct the two positive contractions whose product equal to the given matrix, if they exist.

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1. Introduction

Let M_n be the set of $n \times n$ complex matrices. It is known that every matrix $A \in M_n$ with nonnegative determinant can be written as the product of k positive semidefinite matrices with $k \leq 5$; see [1,2,5] and their references. Moreover, characterizations are given of matrices that can be written as the product of k positive semidefinite matrices but not fewer for $k = 2, \ldots, 5$. In particular, a matrix A is the product of two positive semidefinite matrices if it is similar to a diagonal matrix with nonnegative diagonal entries.

In this paper, characterizations are given to $A \in M_n$ which is a product of two positive contractions, i.e., positive semidefinite matrices with norm not larger than one. Evidently, if a matrix is the product of two positive contractions, then it is a contraction similar to a diagonal matrix with nonnegative diagonal entries. However, the converse is not true. For example, $A = \frac{1}{25}\begin{pmatrix} 9 & 3 \\ 0 & 16 \end{pmatrix}$ is a contraction similar to $\mathrm{diag}\,(9,16)/25$ that is not a product of two positive contractions as shown in [4]. In fact, the result in [4] implies that if $A \in M_n$ is similar to a diagonal matrix with nonzero eigenvalues $a, b \in (0,1]$ then a necessary and sufficient condition for A to be the product of two positive contractions is:

$$\{\|A\|^2 - (a^2 + b^2) + (ab/\|A\|)^2\}^{1/2} \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)};$$

see Corollary 2.6. In particular, a matrix $A = \begin{pmatrix} a & p \\ 0 & b \end{pmatrix} \in M_2$ is the product of two positive contractions if and only if $a, b \in [0,1]$ and $|p| \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)}$.

In Section 2, we will present several characterizations of a square matrix that can be written as the product of two positive (semidefinite) contractions. In Section 3, based on one of the characterizations in Section 2, we use alternating projection method to check the condition and construct the two positive contractions whose product equal to the given matrix if they exist. Some numerical examples generated by Matlab are presented.

2. Characterizations

If A is a product of two positive semidefinite contractions, then A is similar to a diagonal matrix with nonnegative eigenvalues with magnitudes bounded by $\|A\| \leq 1$. We will focus on such matrices in our characterization theorem.

It is known that a matrix A is the product of two orthogonal projections if and only if it is unitarily similar to a matrix which is the direct sum of $I_p \oplus 0_q$ and matrices of the form

$$\begin{pmatrix} a_j & \sqrt{a_j - a_j^2} \\ 0 & 0 \end{pmatrix} \in M_2, \qquad 0 < a_j < 1 \text{ for all } j = 1, \ldots, m;$$

see [3]. Here we give another characterization which will be useful for our study.