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Superregular matrices and applications to convolutional codes ☆

P.J. Almeida, D. Napp, R. Pinto *

CIDMA – Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Aveiro, Portugal

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ABSTRACT

The main results of this paper are twofold: the first one is a matrix theoretical result. We say that a matrix is superregular if all of its minors that are not trivially zero are nonzero. Given a $a \times b$, $a \geq b$, superregular matrix over a field, we show that if all of its rows are nonzero then any linear combination of its columns, with nonzero coefficients, has at least $a - b + 1$ nonzero entries. Secondly, we make use of this result to construct convolutional codes that attain the maximum possible distance for some fixed parameters of the code, namely, the rate and the Forney indices. These results answer some open questions on distances and constructions of convolutional codes posted in the literature [6,9].

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* Corresponding author.
E-mail addresses: palmeida@ua.pt (P.J. Almeida), diego@ua.pt (D. Napp), raquel@ua.pt (R. Pinto).

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1. Introduction

Several notions of superregular matrices (or totally positive) have appeared in different areas of mathematics and engineering having in common the specification of some properties regarding their minors [2,3,5,11,14]. In the context of coding theory these matrices have entries in a finite field $\mathbb{F}$ and are important because they can be used to generate linear codes with good distance properties. A class of these matrices, which we will call *full superregular*, were first introduced in the context of block codes. A full superregular matrix is a matrix with all of its minors different from zero and therefore all of its entries nonzero. It is easy to see that a matrix is full superregular if and only if any $\mathbb{F}$-linear combination of N columns (or rows) has at most $N-1$ zero entries. For instance, Cauchy and nonsingular Vandermonde matrices are full superregular. It is well-known that a systematic generator matrix $G = [I \mid B]^\top$ generates a maximum distance separable (MDS) block code if and only if B is full superregular [13].

Convolutional codes are more involved than block codes and, for this reason, a more general class of superregular matrices had to be introduced. A lower triangular matrix B was defined to be superregular if all of its minors, with the property that all the entries in their diagonals are coming from the lower triangular part of B, are nonsingular, see [6, Definition 3.3]. In this paper, we call such matrices *LT-superregular*. Note that due to such a lower triangular configuration the remaining minors are necessarily zero. Roughly speaking, superregularity asks for all minors that are possibly nonzero, to be nonzero. In [6] it was shown that LT-superregular matrices can be used to construct convolutional codes of rate k/n and degree δ that are strongly MDS provided that $(n-k) \mid \delta$. This is again due to the fact that the combination of columns of these LT-superregular matrices ensures the largest number of possible nonzero entries for any $\mathbb{F}$-linear combination (for this particular lower triangular structure). In other words, it can be deduced from [6] that a lower triangular matrix $B = [b_0 \; b_1 \dots b_{k-1}] \in \mathbb{F}^{n\times k}$, b_i the columns of B, is LT-superregular if and only if for any $\mathbb{F}$-linear combination of columns $b_{i_1}, b_{i_2}, \dots, b_{i_N}$ of B, with $i_j < i_{j+1}$, then $wt(b) \geq wt(b_{i_1}) - N + 1 = (n - i_1) - N + 1$.

It is important to note that in this case due to this triangular configuration it is hard to come up with an algebraic construction of LT-superregular matrices. There exist however two general constructions of these matrices [1,6,7] although they need large field sizes. Unfortunately, LT-superregular matrices allow to construct convolutional codes with optimal distance properties only for certain given parameters of the code. This is because the constant matrix associated to a convolutional code have, in general, blocks of zeros in its lower triangular part. Hence, in order to construct convolutional codes with good distance properties for any set of given parameters a more general notion of superregular matrices needs to be introduced. It is the aim of this paper to do so by generalizing the notion of superregularity to matrices with any structure of zeros. To this end we introduce the notion of *nontrivial* minor (i.e., at least one term in the summation of the Leibniz formula for the determinant is nonzero). Hence, a matrix will be called *superregular* if all of its nontrivial minors are nonzero. This notion naturally