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On the characteristic polynomial of a supertropical adjoint matrix

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ABSTRACT

Let $\chi(A)$ denote the characteristic polynomial of a matrix A over a field; a standard result of linear algebra states that $\chi(A^{-1})$ is the reciprocal polynomial of $\chi(A)$. More formally, the condition $\chi^n(A)\chi^k(A^{-1}) = \chi^{n-k}(A)$ holds for any invertible $n \times n$ matrix A over a field, where $\chi^i(A)$ denotes the coefficient of λ^{n-i} in the characteristic polynomial $\det(\lambda I - A)$. We confirm a recent conjecture of Niv by proving the tropical analogue of this result.

1. Preliminaries

The supertropical semifield is a relatively new concept arisen as a tool for studying problems of tropical mathematics [4]. The supertropical theory is now a developed branch of algebra, and we refer the reader to [5] for a survey of basics and applications. Our arguments make use of some other structures including fields and polynomial rings over them, so it will be convenient for us to work with slightly unusual equivalent description of the supertropical semifield. In particular, we will denote the tropical operations by $\oplus$ and $\odot$ to avoid confusion with standard operations $+$ and $\cdot$ over a field. For the same

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reason, we will use the notation $u^{\odot i}$ in supertropical setting while u^i will denote the power of an element of a field. Similarly, we will denote the supertropical determinant by $\det_\circ$, reserving the notation det for usual determinant over a field. Let us recall the definitions of concepts mentioned above.

Let $(\mathcal{G}, *, \mathbf{1}, \leq)$ be an ordered Abelian group (in multiplicative notation), and $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ be two copies of $\mathcal{G}$. We consider the semiring $\mathcal{S} = \mathcal{G}^{(0)} \cup \mathcal{G}^{(1)} \cup \{\mathbf{0}\}$ with two commutative operations, denoted by $\oplus$ and $\odot$. Assume $i, j \in \{0, 1\}$, $s \in \mathcal{S}$, $a, b \in \mathcal{G}$, and let $a < b$; the operations are defined by $\mathbf{0} \oplus s = s \oplus \mathbf{0} = s$, $\mathbf{0} \odot s = s \odot \mathbf{0} = \mathbf{0}$, $b^{(j)} \oplus a^{(i)} = a^{(i)} \oplus b^{(j)} = b^{(j)}$, $b^{(i)} \oplus b^{(j)} = b^{(0)}$, $a^{(i)} \odot b^{(j)} = (a * b)^{(ij)}$. One can note that $\mathcal{S}$ is isomorphic to the *supertropical semifield*, and the elements from $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ correspond to the *ghost* and *tangible* elements, respectively. In particular, the elements $\mathbf{0}$ and $\mathbf{1}^{(1)}$ are neutral with respect to $\oplus$ and $\odot$, respectively.

We recall that the mapping ν sends $a^{(i)}$ to $a \in \mathcal{G}$ and $\mathbf{0}$ to $\mathbf{0}$; the elements $c, d \in \mathcal{S}$ are *ν-equivalent* whenever $\nu(c) = \nu(d)$, and we write $c \approx_\nu d$ in this case. Also, we write $c \models d$ if either $c = d$ or $c = d \oplus g$, for some ghost element g; this relation is known as the *ghost surpassing* relation, and it is one of fundamental concepts that replaces equality in many theorems taken from classical algebra [5]. By $u^{\odot i}$ we denote the ith supertropical power of u, that is, the result of multiplying u by itself i times.

The operations on $\mathcal{S}$ are now defined, and we can speak of vectors, matrices and polynomials over $\mathcal{S}$; we refer the reader to [2] for a thorough discussion of the matrix and polynomial algebras over a semiring. We define the operations with supertropical polynomials and matrices in the same way as over fields but with standard arithmetic operations replaced by their supertropical counterparts $\oplus$ and $\odot$. In particular, the *supertropical identity matrix* $I_\circ$ has elements $\mathbf{1}^{(1)}$ on the diagonal and $\mathbf{0}$'s everywhere else.

2. The result

Let $A = (a_{ij})$ be a supertropical matrix; its determinant is

$$\det\nolimits_\circ A = \bigoplus_{\sigma \in S_n} a_{1\sigma(1)} \odot \ldots \odot a_{n\sigma(n)},$$

where S_n denotes the symmetric group on $\{1, \ldots, n\}$. The matrix A is said to be *non-singular* if $\det_\circ A$ is tangible; equivalently, A is non-singular if $\det_\circ A$ has a multiplicative inverse in $\mathcal{S}$. The (i, j)th *cofactor* of A is the supertropical determinant of the matrix obtained from A by removing the ith row and jth column. By $\operatorname{adj}_\circ A$ we denote the *adjoint* of A, that is, the $n \times n$ matrix whose (i, j)th entry equals the (j, i)th cofactor. By $\chi_\circ^k(A)$ we denote the coefficient of $\lambda^{\odot(n-k)}$ in the *characteristic polynomial* $\det_\circ(A \oplus \lambda \odot I_\circ)$; for $k \neq 0$, this quantity equals the supertropical sum of all principal $k \times k$ minors of A. The following has been an open problem until now.

Conjecture 1. *(See [1, Conjecture 6.2].) Let $A \in \mathcal{S}^{n \times n}$ be a non-singular matrix. Then, $\chi_\circ^k(\operatorname{adj}_\circ A) \models (\det_\circ A)^{\odot(k-1)} \odot \chi_\circ^{n-k}(A)$ holds for all $k \in \{0, \ldots, n\}$.*