Linear Algebra and its Applications 499 (2016) 66–78

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Small subset sums

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ARTICLE INFO

Article history:
Received 11 February 2015
Accepted 29 February 2016
Available online 14 March 2016
Submitted by R. Brualdi

MSC:
52A40
05B20

Keywords:
Vector sums
Steinitz theorem
Normed spaces

ABSTRACT

Let $\|.\|$ be a norm in $\mathbb{R}^d$ whose unit ball is B. Assume that $V \subset B$ is a finite set of cardinality n, with $\sum_{v \in V} v = 0$. We show that for every integer k with $0 \leq k \leq n$, there exists a subset U of V consisting of k elements such that $\|\sum_{v \in U} v\| \leq \lceil d/2 \rceil$. We also prove that this bound is sharp in general. We improve the estimate to $O(\sqrt{d})$ for the Euclidean and the max norms. An application on vector sums in the plane is also given.

1. Definitions, notation, results

We consider the real d-dimensional vector space $\mathbb{R}^d$ with a norm $\|.\|$ whose unit ball is B. For a finite set $U \subset \mathbb{R}^d$, $|U|$ stands for the cardinality of U, and $s(U)$ for the sum of the elements of U, so $s(U) = \sum_{u \in U} u$, and $s(\emptyset) = 0$ of course.

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http://dx.doi.org/10.1016/j.laa.2016.02.035
0024-3795/

In 1914 Steinitz [12] proved that, in the case of the Euclidean norm, for every finite set $V \subset B$ with $|V| = n$ and $s(V) = 0$, there exists an ordering $v_1, \ldots, v_n$ of the vectors in V such that *all partial sums* have norm at most $2d$, that is

$$\max_{k=1,\ldots,n} \left\| \sum_1^k v_i \right\| \leq 2d.$$

It is important here that the bound $2d$ does not depend on n, the size of V. Steinitz's result implies that for every norm and every finite $V \subset B$ with $s(V) = 0$ there is an ordering along which all partial sums are bounded by a constant that depends only on B. Let $S(B)$ denote the smallest such constant for a given norm with unit ball B, and set $S(d) = \sup S(B)$ where the supremum is taken over all norms in $\mathbb{R}^d$. The best known bounds on $S(d)$ are: $S(d) \leq d$, proved by Sevastyanov [9], and by Grinberg and Sevastyanov [7], and $S(d) \geq \frac{d+1}{2}$, which is shown by an example coming from the ℓ_1 norm [7]. For specific norms, stronger results may hold. In particular, for ℓ_2 and ℓ_∞, it is conjectured that the right order of magnitude of $S(B)$ is $\sqrt{d}$ – although not even $o(d)$ is known.

Steinitz's result immediately implies that for every finite set $V \subset B$ with $s(V) = 0$ and every integer k, $0 \leq k \leq |V|$, there is a subset $U \subset V$ such that $|U| = k$ and $\|s(U)\|$ is not greater than a constant depending only on d, B, k, for instance $S(B)$ is such a constant. Let $T(B, k)$ be the smallest constant with this property, set $T(B) = \sup_k T(B, k)$, and $T(d) = \sup T(B)$ where the supremum is taken over all norms in $\mathbb{R}^d$. It is evident that $T(B, k) \leq k$.

In this paper we investigate $T(B, k)$, $T(B)$ and $T(d)$. Here come our main results. First, the estimate for general norms.

Theorem 1. *Let B be the unit ball of an arbitrary norm on $\mathbb{R}^d$. For any finite set $V \subset B$ with $s(V) = 0$, and for any $k \leq |V|$, there exists a subset $U \subset V$ with k elements, so that*

$$\|s(U)\| \leq \left\lceil \frac{d}{2} \right\rceil.$$

In other words, $T(d) \leq \lceil \frac{d}{2} \rceil$.

Theorem 2. *For every $d \geq 1$, there exists a norm in $\mathbb{R}^d$ with unit ball B, so that $T(B, k) = \lceil \frac{d}{2} \rceil$ for infinitely many values of k. Also, $T(B, k) = k$ for all $k \leq \lfloor \frac{d}{2} \rfloor$.*

Theorems 1 and 2 imply that $T(d) = \lceil \frac{d}{2} \rceil$ for all integers $d \geq 1$.

One expects that for specific norms better estimates are valid. We have proved this in some cases. The unit ball of the norm ℓ_p^d will be denoted by B_p^d. We have the following results in the cases $p = 1, 2, \infty$.