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The product distance matrix of a tree with matrix weights on its arcs

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Let T be a tree with vertex set $[n] = \{1, 2, \ldots, n\}$. For each $i \in [n]$, let m_i be a positive integer. An ordered pair of two adjacent vertices is called an arc. Each arc (i, j) of T has a weight $W_{i,j}$ which is an $m_i \times m_j$ matrix. For two vertices $i, j \in [n]$, let the unique directed path from i to j be $P_{i,j} = x_0, x_1, \ldots, x_d$ where $d \geqslant 1$, $x_0 = i$ and $x_d = j$. Define the product distance from i to j to be the $m_i \times m_j$ matrix $M_{i,j} = W_{x_0,x_1} W_{x_1,x_2} \cdots W_{x_{d-1},x_d}$. Let $N = \sum_{i=1}^{n} m_i$. The $N \times N$ product distance matrix $\mathbf{D}$ of T is a partitioned matrix whose (i, j)-block is the matrix $M_{i,j}$. We give a formula for $\det(\mathbf{D})$. When $\det(\mathbf{D}) \neq 0$, the inverse of $\mathbf{D}$ is also obtained. These generalize known results for the product distance matrix when either the weights are real numbers, or $m_1 = m_2 = \cdots = m_n = s$ and the weights $W_{i,j} = W_{j,i} = W_e$ for each edge $e = \{i, j\} \in E(T)$.

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1. Introduction

Let $n \geqslant 1$ be an integer, and let $[n] = \{1, 2, \ldots, n\}$. Let T be a tree with vertex set $[n]$. Then for any two vertices $i, j \in [n]$, there is a unique path $P_{i,j}$ between i and j. The length (number of edges) of $P_{i,j}$ is called the distance between i and j and is denoted by $d_{i,j}$. The matrix $\mathsf{D} = (d_{i,j})_{i,j\in[n]}$ is the distance matrix of T. Graham and Pollak [7] showed a very beautiful formula $\det(\mathsf{D}) = (-1)^{n-1}2^{n-2}(n-1)$. The determinant of D only depends on the number of vertices of the tree T, but has nothing to do with the structure of it. Yan and Yeh [9] gave a simple proof of Graham and Pollak's formula. When $n \geqslant 2$, D is invertible and Graham and Lovasz [6] gave a formula for the inverse of D.

As an analogue, the exponential distance matrix of the tree T was considered by researchers. Let $e_1, e_2, \ldots, e_{n-1}$ be all the edges of T. Let $q_1, q_2, \ldots, q_{n-1}$ be commutative indeterminates. For any two vertices $i, j \in [n]$, the exponential distance $e_{i,j}$ between i and j is defined as $\prod_{k\in P_{i,j}} q_k$. Since $q_1, q_2, \ldots, q_{n-1}$ commute with each other, then $e_{i,j} = e_{j,i}$. By convention, let $e_{i,i} = 1$ for each $i \in [n]$. The exponential distance matrix of T is $\mathsf{E} = (e_{i,j})_{i,j\in[n]}$. Bapat and Sivasubramanian [4] gave the formula of the determinant of E, $\det(\mathsf{E}) = \prod_{j=1}^{n-1}(1 - q_j^2)$, and the formula of the inverse of E. Let q be an indeterminate with $q^0 = 1$. When $q_1 = q_2 = \cdots = q_{n-1} = q$, then $e_{i,j} = q^{d_{i,j}}$. This special case was dealt by Bapat, Lat and Pati [2].

For product distance, it is natural to consider noncommutative weights from a ring. Let $s \geqslant 1$ be an integer, then all the $s \times s$ matrices over a commutative ring with identity form a natural example of noncommutative rings. For each $j = 1, 2, \ldots, n-1$, let the edge e_j have an $s \times s$ matrix weight W_j. Let $i, j \in [n]$ be two vertices of the tree T, and let $e_{k_1}, e_{k_2}, \ldots, e_{k_r}$ be the consecutive edges that occur on the unique path $P_{i,j}$ from i to j, where $r \geqslant 1$, then define $M_{i,j} = W_{k_1}W_{k_2}\cdots W_{k_r}$. For each $i \in [n]$, define $M_{i,i}$ to be the $s \times s$ identity matrix. The $ns \times ns$ partitioned matrix M, whose (i, j)-block is $M_{i,j}$, is the product distance matrix of the tree T with matrix weights. Bapat and Sivasubramanian [3] showed the formula of the determinant of M, $\det(\mathsf{M}) = \prod_{j=1}^{n-1}\det(I - W_j^2)$, and the formula of the inverse of M when $\det(\mathsf{M}) \neq 0$. Lemma 1 of [3] should have a little modification as follows. There exist a lower triangular matrix L whose diagonal blocks are identity matrices and an upper triangular matrix U whose diagonal blocks are identity matrices such that the matrix $L\mathsf{M}U$ is a block diagonal matrix with diagonal blocks $I, I - W_1^2, I - W_2^2, \ldots, I - W_{n-1}^2$. This modification is implied by Lemma 1 or Lemma 4 of this paper.

In the context of classical distance, matrix weights were also considered by Bapat in [1], where an analogue of Graham and Pollak's formula is proved. General graphs with matrix weights on its arcs were considered by Sato, Mitsuhashi and Morita [8]. They defined a matrix-weighted L-function of such a graph and gave a determinant expression of it.

In this paper, we consider the product distance matrix of a tree with matrix weights on its arcs, where the matrices over a commutative ring with identity may not be square.

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