Linear Algebra and its Applications 496 (2016) 1–35

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Diffuse scattering on graphs

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ARTICLE INFO

Article history:
Received 15 April 2015
Accepted 8 January 2016
Available online 4 February 2016
Submitted by V. Mehrmann

MSC:
05C85
05C25
05C50
68R
35Q

Keywords:
Graph algorithms
Graphs and groups
Graphs and matrices
Discrete mathematics in relation to computer science
Equations of mathematical physics and other areas of application

ABSTRACT

We formulate and analyze difference equations on graphs analogous to time-independent diffusion equations arising in the study of diffuse scattering in continuous media. Moreover, we show how to construct solutions in the presence of weak scatterers from the solution to the homogeneous (background problem) using Born series, providing necessary conditions for convergence and demonstrating the process through numerous examples. In addition, we outline a method for finding Green's functions for Cayley graphs for both abelian and non-abelian groups. Finally, we conclude with a discussion of the effects of sparsity on our method and results, outlining the simplifications that can be made provided that the scatterers are weak and well-separated.

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http://dx.doi.org/10.1016/j.laa.2016.01.012

1. Introduction

Spectral graph theory is a rich and well-developed theory for both the combinatorial and analytic properties of graphs. The following set-up is generally considered. Let $G = (V, E)$ be a graph with vertex set V and edge set E, and L be the combinatorial Laplacian L, or some suitably rescaled variant [16]. We can then formulate a graph analog of Poisson's equation

$$\begin{cases} (Lu)(x) = f(x), & x \in V \\ u(x) = g(x), & x \in \delta V \end{cases} \tag{1}$$

where δV is the set of boundary vertices, which will be discussed in more detail later, and the functions f and g represent internal and boundary sources, respectively. Equation (1) has been studied both when the edges are equally-weighted and when the edge weighting varies throughout the graph [8,16,17]. In this work, we consider the effect of introducing inhomogeneities on the vertices rather than on the edges, as represented by the addition of a (vertex) potential term to equation (1). We call this problem *diffuse scattering on graphs* because of its analogy to a related problem in the continuous setting, where the vertex potential is often called the *absorption*. A similar problem arises in the study of Schrödinger operators on graphs, see for example [6,7,10–12,40]. In order to develop the necessary foundations to formulate corresponding inverse problems, which will be analyzed in subsequent works, we also study the role of boundary conditions on the solutions. In particular, we consider Dirichlet, Neumann and Robin, or mixed, boundary conditions, which are often employed in the continuous setting.

The graph analog of Poisson's equation is related to the classical problem of resistor networks first studied by Kirchhoff in 1847 [31]. In that setting, one is given a collection of interconnected resistors to which a voltage source is attached at various points [20]. The resulting system can be thought of as a weighted graph, with each edge corresponding to a particular resistor and the vertices representing the connections between them [20]. In the event that all the resistors are identical, the voltage at each point satisfies Poisson's equation on the associated graph [19]. In this setting, one seeks either to map the network, finding its corresponding graph [20], solely by measuring the current or potential at various points in the network. This physical analogy is also employed for graph sparsification [41], as well as in near linear-time solvers for symmetric, diagonally dominant linear systems [21,32,47].

Discrete analogs of PDEs on graphs are not limited to Poisson-type problems and are used extensively in lattice dynamics where we consider the graph analog of the Helmholtz equation [34,44], which arises when considering the Fourier transform of the wave equation. In lattice theory, one problem of particular importance is to examine the propagation of phonons through a crystal in order to determine the size and location of imperfections [34,44].