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The generation of all rational orthogonal matrices in $R^{p,q}$

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A method for generating all rational generalized matrices on indefinite real inner product spaces isomorphic to $R^{p,q}$ is presented. The proposed method is based on the proof of a weak version of the Cartan–Dieudonné theorem, handled using Clifford algebras. It is shown that all rational $\mathcal{B}$-orthogonal matrices in an indefinite inner product space $(\mathcal{X}, \mathcal{B})$ are products of simple matrices with rational entries.

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1. Introduction

Orthogonal matrices with rational coefficients play important role in problems in dynamical systems and control [1] and crystallography [2]. The real algebra of quaternions has been used to generate all 3×3 [3] and 4×4 [4] rational orthogonal matrices and a method to generate all $n \times n$ rational orthogonal matrices was proposed in Ref. [5] using Cayley's formula. Here, we propose a method to find all rational orthogonal matrices in real indefinite inner product spaces, which is based on a proof of the Cartan–Dieudonné theorem [6,7]. The advantage of our approach is not only that it allows to generate these rational orthogonal matrices on spaces with signature (p, q) but also that any orthogonal transformation is decomposed as products of reflections by vectors of $\mathbb{Z}^{p,q}$ [7], thus, in the other way, rational orthogonal matrices are obtained by mapping all the points in $\mathbb{Z}^{p,q}$; the inverse, that any rational (or not) orthogonal transformation is obtained by a product of reflections by vectors in $\mathbb{Z}^{p,q}$ is guaranteed by our proof of the aforementioned theorem.

The Cartan–Dieudonné theorem is a fundamental result in geometry. It states that every orthogonal transformation is the composition of reflections across hyperplanes [8]. Constructive proofs of this theorem has been recently proposed [6,7,9] in real indefinite inner product spaces isomorphic to $\mathbb{R}^{p,q}$ was developed using Clifford algebras [6] and it turned out that the algorithm proposed to find explicitly the rotations that decompose a given orthogonal transformation served also to characterize coincidence lattices or, more particularly, coincidence reflections [10]. Even more, the same algorithm provided insights to conceive a procedure to solve some Diophantine equations [10,11]. Here, we show that a slight modification of the proof in [6] will allow to generate all the rational orthogonal matrices in $\mathbb{R}^{p,q}$. The Clifford geometric product between vectors is used as basic tool to handle reflections.

In Section 2 some basic results concerning real vector spaces equipped with a indefinite symmetric bilinear form, orthogonal transformations in this space and Clifford geometric product are presented. In Section 3 a weak version of the Cartan–Dieudonné theorem in a space isomorphic to $\mathbb{R}^{p,q}$ is stated and proved. Based on this proof, a method for generating all the rational orthogonal matrices in $R^{p,q}$ is presented in Section 4. In Section 5 we present some examples and finally, in Section 6 some conclusions are given.

2. Preliminaries

In this work, we consider real indefinite scalar product spaces, that is, real vectors spaces equipped with a indefinite symmetric bilinear form $\mathcal{B}$. For a short account of these generalized inner product spaces we suggest Ref. [7, Sec. 2] and Ref. [12, Ch. 2] for mode details. To fix notation, we just mention that if $\mathcal{X}$ is a real vector space and $\mathcal{B}$ an indefinite non-degenerate symmetric bilinear form then $(\mathcal{X}, \mathcal{B})$ denotes the generalized inner product space. The following definitions will be required: