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Concavity of certain matrix trace and norm functions. II

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ABSTRACT

We refine Epstein's method to prove joint concavity/convexity of matrix trace functions of Lieb type $\mathrm{Tr}\, f(\Phi(A^p)^{1/2}\Psi(B^q)\Phi(A^p)^{1/2})$ and symmetric (anti-) norm functions of the form $\|f(\Phi(A^p)\,\sigma\,\Psi(B^q))\|$, where Φ and Ψ are positive linear maps, σ is an operator mean, and $f(x^\gamma)$ with a certain power γ is an operator monotone function on $(0,\infty)$. Moreover, the variational method of Carlen, Frank and Lieb is extended to general non-decreasing convex/concave functions on $(0,\infty)$ so that we prove joint concavity/convexity of more trace functions of Lieb type.

1. Introduction

In the present paper we consider two-variable matrix functions

$$F(A,B) = f(\Phi(A^p)^{1/2}\Psi(B^q)\Phi(A^p)^{1/2}), \tag{1.1}$$

$$F(A,B) = f(\Phi(A^p)\,\sigma\,\Psi(B^q)), \tag{1.2}$$

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where A, B are positive definite matrices, p, q are real parameters, Φ, Ψ are (strictly) positive linear maps, σ is an operator mean, and f is a real function on $(0, \infty)$. The problem of our concern is joint concavity/convexity of trace and norm functions of such $F(A, B)$ as above. The problem originated with seminal papers of Lieb [17] and Epstein [10] in 1973. In [17], motivated by a conjecture on Wigner–Yanase–Dyson skew information, Lieb established the so-called Lieb concavity/convexity for the matrix trace function $(A, B) \mapsto \operatorname{Tr} X^* A^p X B^q$, that is a special case of (1.1) when $\Phi = X^* \cdot X$, $\Psi = \mathrm{id}$ and $f(x) = x$. An equivalent reformulation is Ando's matrix concavity/convexity of $(A, B) \mapsto A^p \otimes B^q$ in [1]. On the other hand, in [10] Epstein developed a complex function method using theory of Pick functions, called Epstein's method, to prove concavity of the trace function $A \mapsto \operatorname{Tr}(X^* A^p X)^{1/p}$.

In these years, big progress in the subject matter has been made by several authors. For instance, in [8,9] Carlen and Lieb extensively developed concavity/convexity of the trace functions of the forms $\operatorname{Tr}(X^* A^p X)^s$ and $\operatorname{Tr}(A^p + B^p)^s$ of Minkowski type. Very recently, in [7] they with Frank made the best use of the variational formulas discovered in [9] to obtain concavity/convexity of the trace functions

$$(A, B) \mapsto \operatorname{Tr}(A^{p/2} B^q A^{p/2})^s, \tag{1.3}$$

a special case of the trace functions of (1.1) with $f(x) = x^s$. In our previous papers [12,14] we refined Epstein's complex function method to prove joint concavity/convexity results for the trace functions of (1.1) and for the norm/trace functions of (1.2) in the case $f(x) = x^s$. For additional relevant results see [7,9,14] and references therein. Moreover, it is worth noting that our problem on concavity/convexity of (1.3) also emerges from recent developments of new Rényi relative entropies relevant to quantum information theory. That is closely related to monotonicity of those relative entropies under quantum channels (i.e., completely positive and trace-preserving maps), as mentioned in the last part of [7] (see also [2] and references therein).

The present paper is a continuation of [12,14]. In Sections 2 and 3 we further refine Epstein's method used in [12,14] and prove concavity/convexity theorems for the trace functions of (1.1) and for the symmetric (anti-) norm functions of (1.2) when $f(x^\gamma)$ with a certain power γ is an operator monotone function on $(0, \infty)$. In Section 4 we present a general method to passage from concavity/convexity of symmetric (anti-) norm functions to that of trace functions, and apply it to obtain some general concavity/convexity result for the trace functions of (1.2). In Section 5 we extend the variational method in [9,7] to general non-decreasing convex/concave functions on $(0, \infty)$, which enables us to obtain more concavity/convexity theorems for the trace functions of (1.1). To do this, we provide, in the appendix, some variational formulas for such functions on $(0, \infty)$, which might be of independent interest as a theory of conjugate functions (or the Legendre transform) on $(0, \infty)$.