Linear Algebra and its Applications 496 (2016) 221–245

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Generating hyperbolical rotation matrix for a given hyperboloid

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ARTICLE INFO

Article history:
Received 26 October 2015
Accepted 21 January 2016
Available online 9 February 2016
Submitted by P. Semrl

MSC:
15A63
15A66
53A17
53A35
53B30
70B05
70B10
70E17

Keywords:
Hyperbolical rotation matrix
Hyperbolic split quaternion
Rodrigues formula
g-Cayley rotation matrix
g-Householder transformation

ABSTRACT

Hyperbolic rotation is hyperbolically the motion of a smooth object on general hyperboloids given by $-a_1x^2+a_2y^2+a_3z^2 = \pm\lambda$, $\lambda \in \mathbb{R}^+$. In this paper, we investigate the hyperbolical rotation matrices in order to get the motion of a point about a fixed point or axis on the general hyperboloids by defining the Lorentzian Scalar Product Space $\mathbb{R}^{2,1}_{a_1a_2a_3}$ such that the general hyperboloids are the pseudo-spheres of $\mathbb{R}^{2,1}_{a_1a_2a_3}$. We adapt the Rodrigues, Cayley, and Householder methods to $\mathbb{R}^{2,1}_{a_1a_2a_3}$ and define hyperbolic split quaternions to obtain an hyperbolical rotation matrix.

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http://dx.doi.org/10.1016/j.laa.2016.01.038

1. Introduction

A rotation is a rigid body movement around a fixed point or an axis, which leaves the distance between any two points unchanged. All rotations are also a linear maps form a group under the composition called the *rotation group* of a particular space. We can represent them with the orthonormal matrices, called *rotation matrices*, so that the rotation group corresponds to a special non-abelian orthogonal matrix group, denoted by $\mathbf{SO}(3)$ in 3 dimensional space. Rotation matrices have a lot of application areas in geometry, kinematics, physics, computer graphics, animations, and optimization problems involving the estimation of rigid body transformations and other disciplines. Therefore, studying the rotation matrices is an important problem in mathematics.

The rotation matrices can be obtained by using the various methods such as unit quaternions, the Rodrigues formula, the Cayley formula, and the Householder transformation. Each unit quaternion and unit timelike split quaternion correspond to a rotation matrix by means of a special linear map in the Euclidean and Lorentzian 3-space, respectively [18,19,22,26]. The Rodrigues formula says us that the exponential map $e^{\theta A}$ generates a rotation matrix owing to the fact that this map defined from the Lie algebra $\mathfrak{so}(n)$ of skew-symmetric 3×3 matrices to the Lie group $\mathbf{SO}(3)$ is surjective, where A is a skew symmetric matrix and θ is the rotation angle. In this method, only three numbers are needed to construct a rotation matrix in the Euclidean and Lorentzian 3-space [3,10,12]. The Cayley formula $C = (I + A)(I - A)^{-1}$ gives a rotation matrix without using trigonometric functions, where A is a skew symmetric matrix [4,5,14,20,23]. The Householder transformation gives us a reflection matrix so that we can obtain a rotation matrix by taking the composition of two Householder transformations [1,11,21,25].

In recent times, some authors have formed the various studies related to the rotation matrices. For example, these matrices about arbitrary lightlike axis in Minkowski 3-space were studied in [13] by deriving the Rodrigues' rotation formula and using the corresponding Cayley map. In Euclidean 4-space and Minkowski space–time, they were obtained by means of the corresponding decomposition in [7,17]. On the other hand, the paper [16] examined the elliptical rotation which states the motion of a point on an ellipse or ellipsoid by defining the elliptical product space. The main idea of this paper is to research the hyperbolical rotations occurring on the general hyperboloids $-a_1x^2 + a_2y^2 + a_3z^2 = \pm r^2$ and find the corresponding hyperbolical rotation matrices.

The paper is organized as follows. Firstly, we define an indefinite scalar product space denoted by $\mathbb{R}^{p,q}_{a_1a_2\cdots a_n}$ whose pseudo-spheres are the general hyperboloids $-a_1x_1^2 - a_2x_2^2 + \cdots - a_qx_q^2 + a_{q+1}x_{q+1}^2 + \cdots + a_{p+q}x^{p+q} = \pm r^2$. Then, in section 3, we find the hyperbolical rotation matrix using the Clifford algebra and the classical way in the plane $\mathbb{R}^{1,1}_{a_1a_2}$. Later, we generate the hyperbolical rotation matrices in the Lorentzian scalar product space $\mathbb{R}^{2,1}_{a_1a_2a_3}$ via the Rodrigues, Cayley and Hoseholder's methods. Finally, we construct the hyperbolic split quaternions and give some properties of this quaternions so that each H-timelike hyperbolic split quaternion corresponds to an hyperbolical rotation matrix in $\mathbb{R}^{2,1}_{a_1a_2a_3}$.