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Some inequalities for central moments of matrices ☆

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ABSTRACT

In this paper we shall study non-commutative central moment inequalities with a focus on whether the commutative bounds are tight in the non-commutative case as well. We prove that the answer is affirmative for the fourth central moment and several particular results are given in the general case. As an application, we shall present some lower estimates of the spread of Hermitian and normal matrices as well.

1. Introduction

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$. Then its pth central moments, $1 \leq p < \infty$, are defined by the formula

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$$\mu_p(X) = \int_\Omega \left| X - \int_\Omega X\, dP \right|^p dP.$$

The most studied non-commutative analogue of these quantities is the non-commutative variance or quantum variance. Let $M_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices. Whenever $\Phi\colon M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is a positive unital linear map, the variance of a matrix A can be defined as $\Phi(A^*A) - \Phi(A)^*\Phi(A)$. Several interesting properties of this variance can be found in Bhatia's book [4]. For instance, special choices of Φ and applications of variance estimates provided simple new proofs of spread estimates of normal and Hermitian matrices as well, see [5] and [6]. On the other hand, the first sharp estimate of the non-commutative variance appeared in K. Audenaert's paper [1] in connection with the Böttcher–Wenzel commutator estimate. For several different proofs of his result, we refer to [9,6,23]. Recently extremal properties of the quantum variance were studied in [20].

It is simple to see that if ω is a state (i.e. positive linear functional of norm 1) of the algebra $M_n(\mathbb{C})$, then one has the upper bound

$$\omega(|A - \omega(A)|^2) = \omega(|A|^2) - |\omega(A)|^2 \le \|A\|^2$$

(see [5, Theorem 3.1] for positive linear maps). A careful look of the previous inequality says that the non-commutative variance cannot be larger than the ordinary variance of random variables. In fact, if X is a Bernoulli variable, that is, $P(X = 0) = p$ and $P(X = 1) = 1 - p$ $(0 \le p \le 1)$, then $\mu_2(X) \le 1/4$ holds. Furthermore, for any (complex-valued) random variable $Z\colon \Omega \to \mathbb{C}$ the inequality

$$\begin{aligned}\sqrt{\mu_2(Z)} &\le 2 \max\{\sqrt{\mu_2(X)}\colon X \text{ Bernoulli random variable }\}\|Z\|_\infty \\ &= \|Z\|_\infty\end{aligned}$$

readily follows, see e.g. [1, Theorem 7] and [15, Theorem 2]. In addition, one can have the following upper bound for the nth $(n \in \mathbb{N})$ central moment of a normal element A in matrix and C^*-algebras:

$$\sqrt[n]{\omega(|A - \omega(A)|^n)} \le 2 \max\{\sqrt[n]{\mu_n(X)}\colon X \text{ Bernoulli random variable }\} \min_{\lambda \in \mathbb{C}} \|A - \lambda\|,$$

see [15, Theorem 2].

Our main motivation is to provide sharp upper bounds on the non-commutative central moments of arbitrary matrices and to prove that the non-commutative dispersion cannot be larger than that of the commutative one. We note that a very similar phenomenon was observed by K. Audenaert [2]. He proved that the asymmetry of the quantum relative entropy essentially cannot be larger than that of two Bernoulli distributions. Now we are able to tackle the moment problem for $1 \le p \le 2$, $p = 4$ and for