Linear Algebra and its Applications 496 (2016) 288–306

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the diagonalizability of a matrix by a symplectic equivalence, similarity or congruence transformation

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A R T I C L E   I N F O

Article history:
Received 22 November 2015
Accepted 18 January 2016
Available online 10 February 2016
Submitted by R. Brualdi

MSC:
15A21
15A23

Keywords:
Symplectic matrix
Diagonalizable
Takagi factorization
Singular Value Decomposition

A B S T R A C T

A symplectic matrix $S \in \mathbb{C}^{2n\times 2n}$ satisfies $S = J^{-1}S^TJ$ for $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$. We will consider symplectic equivalence, similarity and congruence transformations and answer the question under which conditions a $2n \times 2n$ matrix is diagonalizable under one of these transformations. In particular, we will give symplectic analogues of the singular value decomposition and the Takagi factorization.

1. Introduction

It is well known that

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http://dx.doi.org/10.1016/j.laa.2016.01.030

- Any matrix $A \in \mathbb{C}^{n\times n}$ is unitarily equivalent to a diagonal matrix (singular value decomposition (SVD) [6, Section 3.4]).
- A matrix $A \in \mathbb{C}^{n\times n}$ is unitarily similar to a diagonal matrix if and only if A is normal (that is, $AA^* = A^*A$ where A^* denotes the conjugate transpose of A [6, Theorem 2.5.4]).
- A matrix $A \in \mathbb{C}^{n\times n}$ is unitarily congruent to a diagonal matrix if and only if A is symmetric (Takagi factorization [6, Corollary 4.4.4]).

Note that unitary matrices are those matrices which preserve length in the Euclidean norm. That is, those matrices $U \in \mathbb{C}^{n\times n}$ satisfying $[Ux, Uy] = [x, y]$ for all $x, y \in \mathbb{C}^n$ where $[\cdot,\cdot]$ denotes the Euclidean inner product on $\mathbb{C}^n$

$$[x, y] = x^*y.$$

Observe that U is unitary if and only if $UU^* = U^*U = I$ and that a unitary matrix is necessarily normal. Clearly, the adjoint of a matrix $A \in \mathbb{C}^{n\times n}$ with respect to $[\cdot,\cdot]$ is given by A^*.

Here we will discuss symplectic analogues of the above statements. The symplectic matrices are the isometries of the bilinear form $\langle\cdot,\cdot\rangle_{J_{2n}}$ associated with $J_{2n} \equiv \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$

$$\langle x, y\rangle_{J_{2n}} \equiv x^T J_{2n} y$$

for all $x, y \in \mathbb{C}^{2n}$ (that is, any matrix P with $\langle Px, Py\rangle_{J_{2n}} = \langle x, y\rangle_{J_{2n}}$, for all $x, y \in \mathbb{C}^{2n}$ is symplectic). Observe that $A \in \mathbb{C}^{2n\times 2n}$ is symplectic if and only if $A^T J_{2n} A = J_{2n}$. Clearly, the adjoint of a matrix $A \in \mathbb{C}^{2n\times 2n}$ with respect to $\langle\cdot,\cdot\rangle_{J_{2n}}$ is given by $J_{2n}^{-1} A^T J_{2n}$. We omit the subscript if the size of J is clear from the context. For the ease of notation, let us define the operator $\phi_J : \mathbb{C}^{2n\times 2n} \to \mathbb{C}^{2n\times 2n}$ by

$$\phi_J(A) = J^{-1} A^T J. \tag{1}$$

In analogy to the definition of normal matrices, a matrix $A \in \mathbb{C}^{2n\times 2n}$ will be called ϕ_J *normal* if

$$\phi_J(A)A = A\phi_J(A). \tag{2}$$

In particular, we will prove

- A matrix $A \in \mathbb{C}^{2n\times 2n}$ is symplectically equivalent to a diagonal matrix if and only if $A\phi_J(A)$ is similar to $\phi_J(A)A$ and $A\phi_J(A)$ is diagonalizable (symplectic SVD).
- A matrix $A \in \mathbb{C}^{2n\times 2n}$ is symplectically similar to a diagonal matrix if and only if $\phi_J(A)A = A\phi_J(A)$ (that is, A is ϕ_J normal) and A is diagonalizable.
- A matrix $A \in \mathbb{C}^{2n\times 2n}$ is symplectically congruent to a diagonal matrix if and only if A is symmetric and $A\phi_J(A)$ is diagonalizable (symplectic Takagi factorization).