Linear Algebra and its Applications 496 (2016) 351–380

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Constrained evolution algebras and dynamical systems of a bisexual population

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ARTICLE INFO

Article history:
Received 9 October 2015
Accepted 4 January 2016
Available online 10 February 2016
Submitted by R. Brualdi

MSC:
17D99
60J10

Keywords:
Bisexual population
Evolution algebra
Evolution operator
Fixed point
Limit point

ABSTRACT

Consider a bisexual population such that the set of females can be partitioned into finitely many different types indexed by $\{1, 2, \dots, n\}$ and, similarly, that the male types are indexed by $\{1, 2, \dots, \nu\}$. Recently an evolution algebra of bisexual population was introduced by identifying the coefficients of inheritance of a bisexual population as the structure constants of the algebra. In this paper we study constrained evolution algebra of bisexual population in which type "1" of females and males have preference. For such algebras sets of idempotent and absolute nilpotent elements are known. We consider two particular cases of this algebra, giving more constraints on the structural constants of the algebra. By the first our constraint we obtain an $n + \nu$-dimensional algebra with a matrix of structural constants containing only 0 and 1. In the second case we consider $n = \nu = 2$ but with general constraints. In both cases we study dynamical systems generated by the quadratic evolution operators of corresponding constrained algebras. We find all fixed points, limit points and some 2-periodic points of the dynamical systems. Moreover we study several properties of the constrained algebras connecting them to the dynamical systems. We give some biological interpretation of our results.

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http://dx.doi.org/10.1016/j.laa.2016.01.048

1. Introduction

Population genetics is the study of the *distributions or states* and *changes* of allele frequency in a population, as the population is subject to the four main evolutionary processes: natural selection, genetic drift, mutation and gene flow. It also takes into account the factors of recombination, population subdivision and population structure.

In this paper we study of evolution algebras and dynamical systems (of states) of sex linked populations. Depending on the matrix of structural constants these algebras are renamed by several distinct names. For example, in mathematical genetics, a genetic algebra is a (possibly non-associative) algebra used to model inheritance in genetics. Some variations of these algebras are called train algebras, special train algebras, gametic algebras, Bernstein algebras, copular algebras, zygotic algebras and baric algebras (also called weighted algebra). The study of these algebras was started by Etherington [4].

In applications to genetics these algebras often have a basis corresponding to the genetically different gametes and the structure constants of the algebra encode the probabilities of producing offspring of various types. The laws of inheritance are then encoded as algebraic properties of the algebra.

For surveys of genetic algebras see [1,10,12] and [17].

In [15] an evolution algebra (EA) was introduced (denoted by E), which is an algebra over a field with a basis on which multiplication is defined by the product of distinct basis terms being zero. This EA is commutative, but not necessarily associative or power-associative [15]. Under some conditions on the matrix of structural constants the algebra E is a baric algebra [2].

In [10] an EA (denoted by $\mathcal{A}$) associated to the free population is introduced and using this non-associative algebra many results are obtained in explicit form, e.g. the explicit description of stationary quadratic operators and the explicit solutions of a nonlinear evolutionary equation in the absence of selection, as well as general theorems on convergence to equilibrium in the presence of selection. Note that the algebra $\mathcal{A}$ is a baric algebra.

Recently, in [9] a bisexual population was considered and an EA (denoted by $\mathcal{B}$) using inheritance coefficients of the population was introduced. This algebra is a natural generalization of the algebra $\mathcal{A}$ of free population. Moreover, the evolution algebra of a bisexual population $\mathcal{B}$ is different from the EA E defined in [15]. In fact, the table of multiplications of E *non-zero* multiplications is comprised of *only* the square of each basis element. However in $\mathcal{B}$ the square of *each* basis element is zero. Note also that the algebra $\mathcal{B}$ is never baric, but it is a dibaric algebra.

In [16] a notion of gonosomal algebra is introduced. This algebra extends the evolution algebra of the bisexual population $\mathcal{B}$. It is shown that gonosomal algebras represent algebraically a wide variety of sex determination systems observed in bisexual populations. Moreover, it was shown that unlike $\mathcal{B}$ the gonosomal algebra is not dibaric, in general.

Note that each multiplication table of an n-dimensional algebra A (n can be infinite) over a field K generates an operator V from K^n to itself. The properties of the algebras