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Sets of matrices with singleton spectra generated by positive matrices

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In this paper we present order versions of Kaplansky's results on triangularizability of semigroups and Lie sets of linear operators with singleton spectra. The Jordan set case is more difficult to deal with. For the Jordan set $\mathcal{J}$ of positive matrices with singleton spectra we prove the following:

(a) If $\mathcal{J}$ contains a nilpotent matrix N with $N^2 \neq 0$, then $\mathcal{J}$ is decomposable.
(b) If $\mathcal{J}$ does not contain nilpotent matrices or if $\mathcal{J}$ consists of nilpotent matrices, then $\mathcal{J}$ is completely decomposable.

We also give an example of an irreducible Jordan set of positive matrices with singleton spectra which contains a nonzero square-zero matrix. Furthermore, we determine the lower bound for the number of standard subspaces invariant under a given Jordan set of positive matrices with singleton spectra which contains a nilpotent matrix.

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1. Introduction

Schur's theorem states that every linear operator on a finite-dimensional vector space over an algebraically closed field is triangularizable. The natural question that arises here is which algebraic structures share the same property, i.e., which sets of linear operators are simultaneously triangularizable. Apart from the commutative ones, Jacobson [7] obtained the following impressive result.

Theorem 1.1 *(Jacobson's theorem). Let $\mathcal{N}$ be a set of nilpotent linear operators on a finite-dimensional vector space. Assume that for each pair A, B in $\mathcal{N}$ there exists a scalar c such that $AB - cBA$ belongs to $\mathcal{N}$. Then $\mathcal{N}$ is triangularizable.*

Jacobson's theorem can be applied to prove that semigroups, Lie sets and Jordan sets of nilpotent matrices are triangularizable. Triangularizability of semigroups of nilpotent linear operators on finite-dimensional vector spaces was first obtained by Levitzki [11]. The special case of Jacobson's theorem for Lie sets of nilpotent linear operators also remarkably strengthens the celebrated Engel's theorem about triangularizability of Lie algebras of nilpotent linear operators on a finite-dimensional vector space. For a generalization of Jacobson's result we refer the reader to [14].

Somewhat parallel result to the Levitzki's result was obtained by Kolchin [10]. He proved triangularizability of semigroups whose elements are unipotent linear operators. Kaplansky [8, Th. H., p. 137] simultaneously unified Levitzki's and Kolchin's result to the case of semigroups of linear operators with singleton spectra. For a recent extension of Kaplansky's theorem in the case of the field of complex numbers we refer the reader to [16].

Triangularizability of Lie sets of linear operators with singleton spectra was also obtained by Kaplansky [9, Theorem 4]. Jordan algebras over a field Φ of characteristic different from 2 of linear operators with singleton spectra were considered by Albert [2] (see also [12]). He proved that for such a Jordan algebra $\mathcal{J}$ containing the identity I we have $\mathcal{J} = \Phi \cdot I + \mathcal{N}$ where $\mathcal{N}$ is a nil-Jordan subalgebra of $\mathcal{J}$. Since Jordan sets of nilpotent linear operators on a finite-dimensional vector space are triangularizable, we have that $\mathcal{J}$ is triangularizable. It should be noted that there exist irreducible Jordan sets of linear operators with singleton spectra (Example 3.5).

The main purpose of this paper is to present order analogs that hold for semigroups, Lie and Jordan sets of matrices with singleton spectra, generated by positive matrices.

2. Preliminaries

It is well-known that the n-dimensional vector space $\mathbb{R}^n$ ordered componentwise is a vector lattice. Vectors with nonnegative components are called *positive vectors*. The *absolute value* $|x| = \sup\{x, -x\}$ of the vector $x = (x_1, \ldots, x_n)$ is the vector $(|x_1|, \ldots, |x_n|)$. A linear subspace J of $\mathbb{R}^n$ is called a *standard subspace* whenever there exists a subset