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Quasi-permutation singular matrices are products of idempotents

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ABSTRACT

A matrix $A \in M_n(R)$ with coefficients in any ring R is a quasi-permutation matrix if each row and each column has at most one nonzero element. It is shown that a singular quasi-permutation matrix with coefficients in a domain is a product of idempotent matrices. As an application, we prove that a nonnegative singular matrix having nonnegative von Neumann inverse (also known as generalized inverse) is a product of nonnegative idempotent matrices.

1. Introduction and preliminaries

Initiated by Erdos (cf. [6]) the problem of decomposing singular matrices into a product of idempotent matrices has been intensively studied by several authors (cf. Fountain [7], Hannah O'Meara [8] and also [1,3,5,12]). Recently it has been shown in [2] that

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for $n \geq 1$, every $n \times n$ nonnegative singular matrix $A \in M_n(\mathbb{R})$ of rank one has a decomposition into a product of at most three nonnegative idempotent matrices.

A matrix $A \in M_n(R)$ with coefficients in a ring R, is called a quasi-permutation matrix if each row and each column has at most one nonzero element. Using combinatorial techniques, we show that singular quasi-permutation matrices with coefficients in any domain can always be represented as a product of idempotent matrices (Theorem 7).

As an application, we show that nonnegative matrices having nonnegative von Neumann inverse (also known as generalized inverse) can be decomposed into a product of nonnegative idempotents (Theorem 15). Indeed, the well-known structure of nonnegative idempotent matrices ([4], p.65) and the structure of nonnegative matrices that have a nonnegative von Neumann inverse reveal strong links with rank one matrices and the quasi-permutation matrices (cf. [11]). We make use of their structure to establish our results. For ring theory terminologies and definitions the reader may refer to [9,13].

For convenience, we state below two lemmas that are used often in the proofs of our results [2].

Lemma 1. *Any singular nonnegative matrix of rank 1 can be presented as a product of three nonnegative idempotent matrices of rank one.*

Lemma 2. *Any nonnegative nilpotent matrix is a product of nonnegative idempotent matrices.*

2. Quasi-permutation matrices

Let us recall that for a permutation $\sigma \in S_n$, the permutation matrix P_σ associated with σ is an $n \times n$ matrix defined by

$$P_\sigma = \sum_{i=1}^{n} e_{i,\sigma(i)}.$$

We consider a more general situation as given in the following definition.

Definition 3. A matrix $A \in M_n(R)$ with coefficients in a ring R will be called a quasi-permutation matrix if each row and each column has at most one nonzero element.

Remarks 4.

(a) A quasi-permutation matrix can be singular and, in this case, it has at least one zero row and one zero column. We will mainly work with rows but the analogous properties for columns also hold (acting on the right with given permutation matrices).
(b) Particularly important for our purposes will be the quasi-permutation matrices, denoted by $P_{\sigma,l}$, $\sigma \in S_n$, $l \in \{1, \dots, n\}$ obtained from P_σ by changing the nonzero element of the lth row of P_σ to 0. We thus have