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Counting zero kernel pairs over a finite field

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ABSTRACT

Helmke et al. have recently given a formula for the number of reachable pairs of matrices over a finite field. We give a new and elementary proof of the same formula by solving the equivalent problem of determining the number of so called zero kernel pairs over a finite field. We show that the problem is equivalent to certain other enumeration problems and outline a connection with some recent results of Guo and Yang on the natural density of rectangular unimodular matrices over $\mathbb{F}_q[x]$. We also propose a new conjecture on the density of unimodular matrix polynomials.

1. Introduction

Let $\mathbb{F}_q$ denote the finite field with q elements. For positive integers n, k, we denote by $M_{n,k}(\mathbb{F}_q)$ the set of all $n \times k$ matrices with entries in $\mathbb{F}_q$ and by $M_n(\mathbb{F}_q)$ the set of all square $n \times n$ matrices with entries in $\mathbb{F}_q$. Throughout this paper we assume $k < n$ unless otherwise stated. Consider the following problems:

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Problem 1.1. How many matrices in $M_{n,k}(\mathbb{F}_q)$ occur as the submatrix formed by the first k columns of some matrix in $M_n(\mathbb{F}_q)$ with irreducible characteristic polynomial?

Problem 1.2. How many matrices $Y \in M_{n,k}(\mathbb{F}_q)$ have the property that the linear matrix polynomial

$$x\begin{bmatrix} I_k \\ \mathbf{0} \end{bmatrix} - Y \tag{1}$$

is unimodular (i.e. has all invariant factors equal to 1)? (I_k denotes the $k \times k$ identity matrix, $\mathbf{0}$ denotes the $n-k \times k$ zero matrix.)

Problem 1.3. How many pairs of matrices $(A, B) \in M_k(\mathbb{F}_q) \times M_{k,n-k}(\mathbb{F}_q)$ have the property that

$$\operatorname{rank}\begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix} = k? \tag{2}$$

Problem 1.4. If V is an n-dimensional vector space over $\mathbb{F}_q$ and W is a fixed k-dimensional subspace of V, how many linear transformations $T : W \to V$ have the property that the only T-invariant subspace (contained in W) is the zero subspace?

Interestingly, all the above problems are equivalent and have the same answer given by $\prod_{i=1}^{k}(q^n - q^i)$. Problem 1.3 was considered by Kocięcki and Przyłuski [8] in the context of estimating the proportion of reachable linear systems over a finite field. They gave an explicit answer to Problem 1.3 in the cases $n - k = 1, 2$. In the same paper, they stated that the general problem “seems to be rather difficult”. In fact, the problem of finding an explicit formula for the number of ‘reachable pairs’ (A, B) (i.e. pairs of matrices satisfying (2)) has been settled only very recently by Helmke et al. [7, Thm. 1]. The proof relies on some earlier results by Helmke [5,6] and uses some advanced geometric techniques.

In this paper, we give a new proof of the same formula for the number of reachable pairs by first showing in Section 2 that Problems 1.1, 1.2, 1.3 above are indeed equivalent. In Section 3 we explain the connection with Problem 1.4 which we subsequently solve. Our proof is self-contained and uses only elementary methods in linear algebra and the q-Vandermonde identity for Gaussian binomial coefficients. We also highlight a connection between Problem 1.2 and some recent results by Guo and Yang [4] on the natural density of rectangular unimodular matrices over $\mathbb{F}_q[x]$. We then propose a new conjecture on the density of unimodular matrices that generalizes the problems stated earlier in the introduction.

2. Background

Problem 1.1 is in fact a combinatorial matrix completion problem – we would like to count the number of matrices in $M_{n,k}(\mathbb{F}_q)$ for which there exists a completion (by padding