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On the inverse of a tensor

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ABSTRACT

In this paper, we consider the left (right) inverse of a tensor. We characterize the existence of any order k left (right) inverse of a tensor, and show the expression of left (right) inverse of a tensor. We also present a result for the similarity of tensors.

1. Introduction

Let $\mathbb{C}$ be the complex field. For a positive integer n, let $\langle n \rangle = \{1, \cdots, n\}$. An order m tensor consists of $n_1 \times \cdots \times n_m$ entries in $\mathbb{C}$:

$$\mathcal{A} = (a_{i_1 \cdots i_m}), a_{i_1 \cdots i_m} \in \mathbb{C}, i_j \in \langle n_j \rangle, j = 1, \cdots, m.$$

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Sometimes, we write $a_{i_1\cdots i_m}$ as $a_{i_1\alpha}$, where $\alpha = i_2\cdots i_k$. When $m=2$, $\mathcal{A}$ is an $n_1\times n_2$ matrix. If $n_1=\cdots=n_m=n$, $\mathcal{A}$ is called an order m dimension n tensor. The set of all order m dimension n tensors is denoted by $\mathbb{C}^{[m,n]}$.

Now we introduce the following product of tensors.

Definition 1.1. (See [3].) Let $\mathcal{A}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_2}$ and $\mathcal{B}\in\mathbb{C}^{n_2\times\cdots\times n_{k+1}}$ be two tensors of order $m(\geq 2)$ and $k(\geq 1)$, respectively. The product $\mathcal{A}\circ\mathcal{B}$ is the tensor $\mathcal{C}$ of order $(m-1)(k-1)+1$ with entries:

$$c_{j\alpha_2\cdots\alpha_m}=\sum_{j_2,\cdots,j_m=1}^{n_2}(a_{jj_2\cdots j_m}\prod_{i=2}^{m}b_{j_i\alpha_i}),$$

where $j\in\langle n_1\rangle$, $\alpha_2,\cdots,\alpha_m\in\langle n_3\rangle\times\cdots\times\langle n_{k+1}\rangle$.

The tensor product defined in Definition 1.1 has the following properties [3,1]:

(1) $(\mathcal{A}_1+\mathcal{A}_2)\circ\mathcal{B}=\mathcal{A}_1\circ\mathcal{B}+\mathcal{A}_2\circ\mathcal{B}$, where $\mathcal{A}_1,\mathcal{A}_2\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_2},\mathcal{B}\in\mathbb{C}^{n_2\times\cdots\times n_{k+1}}$.

(2) $A\circ(\mathcal{B}_1+\mathcal{B}_2)=A\circ\mathcal{B}_1+A\circ\mathcal{B}_2$, where $A\in\mathbb{C}^{n_1\times n_2},\mathcal{B}_1,\mathcal{B}_2\in\mathbb{C}^{n_2\times\cdots\times n_{k+1}}$.

(3) $\mathcal{A}\circ I_{n_2}=\mathcal{A}, I_{n_2}\circ\mathcal{B}=\mathcal{B}$, where $\mathcal{A}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_2},\mathcal{B}\in\mathbb{C}^{n_2\times\cdots\times n_{k+1}}$, I_{n_2} is the identity matrix of order n_2.

(4) $(\mathcal{A}\circ\mathcal{B})\circ\mathcal{C}=\mathcal{A}\circ(\mathcal{B}\circ\mathcal{C})$, where $\mathcal{A}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_2},\mathcal{B}\in\mathbb{C}^{n_2\times n_3\times\cdots\times n_3},\mathcal{C}\in\mathbb{C}^{n_3\times\cdots\times n_r}$.

Let $\mathcal{D}=(d_{i_1\cdots i_m})\in\mathbb{C}^{[m,n]}$. We call the entry $d_{i\cdots i}$ a diagonal entry of a tensor $\mathcal{D}$, $i=1,\cdots,n$. A tensor $\mathcal{D}$ is said to be diagonal if all its non-diagonal entries are zero. The unit tensor $\mathcal{I}=(\delta_{i_1\cdots i_m})$ of order m dimension n is a special diagonal tensor, i.e., its entries satisfy the following equation:

$$\delta_{i_1\cdots i_m}=\begin{cases}1 & , \quad i_1=\cdots=i_m\\ 0 & , \quad else\end{cases}.$$

Sometimes, we denote the unit tensor of order m by $\mathcal{I}_m$.

Definition 1.2. (See [3].) Let $\mathcal{A}\in\mathbb{C}^{[m,n]},\mathcal{B}\in\mathbb{C}^{[k,n]}$. If $\mathcal{A}\circ\mathcal{B}=\mathcal{I}$, then $\mathcal{A}$ is called an order m left inverse of $\mathcal{B}$, and $\mathcal{B}$ is called an order k right inverse of $\mathcal{A}$.

In [3], the authors presented some results for order 2 left (right) inverse of tensors. In this paper, we consider the order k left (right) inverse of a tensor, our results extend the corresponding ones in [3]. The contribution of this paper is given below:

- Conclude that $\mathcal{A}$ has left (right) inverse of order 2 if and only if $\mathcal{A}$ has left (right) inverse of order k;
- Give the expression of order k left (right) inverse of $\mathcal{A}$;
- Give a result between reversibility and similarity of tensors.