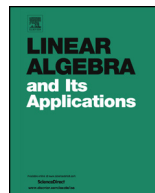




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# Partial isometries and pseudoinverses in semi-Hilbertian spaces



Guillermina Fongí<sup>a,1</sup>, M. Celeste Gonzalez<sup>a,b,\*,1</sup>

<sup>a</sup> Instituto Argentino de Matemática “Alberto P. Calderón”, Saavedra 15, Piso 3 (1083), Buenos Aires, Argentina

<sup>b</sup> Instituto de Ciencias, Universidad Nacional de General Sarmiento, Argentina

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## ABSTRACT

In this article the concepts of partial isometries, normal partial isometries and generalized projections in the context of operators defined on a semi-Hilbertian space are developed. In particular, we analyze the relationship between these operators and different notions of pseudoinverses in semi-Hilbertian spaces. Finally, we apply the results obtained to describe the set of weighted projections into closed subspaces.

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## 1. Introduction

Let  $\mathcal{H}$  be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and  $L(\mathcal{H})$  the algebra of bounded linear operators on  $\mathcal{H}$ . If  $A \in L(\mathcal{H})$  is positive (semi-definite) and  $\langle \cdot, \cdot \rangle_A$  is the semi-inner product on  $\mathcal{H}$  defined by  $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$  for all  $\xi, \eta \in \mathcal{H}$  then  $(\mathcal{H}, \langle \cdot, \cdot \rangle_A)$  is called

\* Corresponding author at: Instituto Argentino de Matemática “Alberto P. Calderón”, Saavedra 15, Piso 3 (1083), Buenos Aires, Argentina.

E-mail addresses: [gfongi@conicet.gov.ar](mailto:gfongi@conicet.gov.ar) (G. Fongí), [celeste.gonzalez@conicet.gov.ar](mailto:celeste.gonzalez@conicet.gov.ar) (M.C. Gonzalez).

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a semi-Hilbertian space. In the literature there are many papers that study operators defined on semi-Hilbertian spaces. The first work on this subject is due to Krein [15]. There, the author deals with operators which are  $A$ -selfadjoint, i.e., which are selfadjoint respect to the semi-inner product  $\langle \cdot, \cdot \rangle_A$ . In [10,11] the existence of  $A$ -selfadjoint idempotents with a fixed closed range is studied. Later, in [3] operators which are isometric, unitary and partially isometric under the structure induced by  $\langle \cdot, \cdot \rangle_A$  are described. One of the main characteristics of the operators defined in  $(\mathcal{H}, \langle \cdot, \cdot \rangle_A)$  is that the existence of an adjoint operator for  $\langle \cdot, \cdot \rangle_A$  is not guaranteed. Therefore, the extension of certain properties of operators in  $L(\mathcal{H})$  to bounded linear operators defined on a semi-Hilbertian space is not trivial.

In this article we continue with the study of partial isometries in the context of semi-Hilbertian spaces that began in [3] and followed in [2]. Recall that  $T \in L(\mathcal{H})$  is a partial isometry if  $\|T\xi\| = \|\xi\|$  for all  $\xi \in N(T)^\perp$ , where  $N(T)$  denotes the nullspace of  $T$ . The following equivalent conditions are well-known for  $T \in L(\mathcal{H})$ :

1.  $T$  is a partial isometry;
2.  $T^*T$  is an idempotent operator;
3.  $TT^*$  is an idempotent operator;
4.  $TT^*T = T$ ;
5.  $T^*TT^* = T^*$ ;
6.  $T^* = T^\dagger$ ;
7.  $T^*\eta$  is the unique least square solution with minimal norm of the equation  $T\xi = \eta$  for all  $\xi \in \mathcal{H}$ ;

where  $T^*$  and  $T^\dagger$  denote the adjoint operator of  $T$  and the Moore–Penrose inverse of  $T$ , respectively.

Given a positive operator  $A \in L(\mathcal{H})$ , an operator  $T \in L(\mathcal{H})$  is an  $A$ -partial isometry if  $\|T\xi\|_A = \|\xi\|_A$  for all  $\xi \in N(AT)^\perp_A$ ; where  $\|\xi\|_A = \langle \xi, \xi \rangle_A^{1/2}$  is the seminorm on  $\mathcal{H}$  induced by  $A$  and  $N(AT)^\perp_A$  denotes the orthogonal complement of  $N(AT)$  with respect to  $\langle \cdot, \cdot \rangle_A$ . However this definition does not allow to ensure that an  $A$ -partial isometry admits an  $A$ -adjoint operator. Therefore, it is not trivial how to extend the equivalences 1 to 7 above to  $A$ -partial isometries.

One of the main goals of this article is to analyze whether the equivalences stated above for partial isometries are still valid in the context of semi-Hilbertian spaces. For this purpose we deal with  $A$ -partial isometries that admit  $A$ -adjoint and we fix a distinguished one. There is a preliminary study in this direction. In [3] an equivalence like  $1 \leftrightarrow 2$  above is shown in the context of semi-Hilbertian spaces under different hypotheses, for example the existence of an  $A$ -selfadjoint idempotent with a fixed closed range; or the closedness of the range of  $A$ . Later, in [2] the relationship between  $A$ -partial isometries and generalized inverses is investigated. In particular, the connection between  $A$ -partial isometries and  $A$ -generalized inverses is described. Here, the  $A$ -generalized inverses will play the roll of the Moore–Penrose inverses in the semi-Hilbertian space  $(\mathcal{H}, \langle \cdot, \cdot \rangle_A)$ . This study is related to equivalences  $1 \leftrightarrow 4 \leftrightarrow 6$  above. In Proposition 3.4 we study equiva-

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