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Partial isometries and pseudoinverses in semi-Hilbertian spaces

Guillermina Fongi [a,1], M. Celeste Gonzalez [a,b,*,1]

[a] *Instituto Argentino de Matemática "Alberto P. Calderón", Saavedra 15, Piso 3 (1083), Buenos Aires, Argentina*
[b] *Instituto de Ciencias, Universidad Nacional de General Sarmiento, Argentina*

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ABSTRACT

In this article the concepts of partial isometries, normal partial isometries and generalized projections in the context of operators defined on a semi-Hilbertian space are developed. In particular, we analyze the relationship between these operators and different notions of pseudoinverses in semi-Hilbertian spaces. Finally, we apply the results obtained to describe the set of weighted projections into closed subspaces.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\ ,\ \rangle$ and $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. If $A \in L(\mathcal{H})$ is positive (semi-definite) and $\langle\ ,\ \rangle_A$ is the semi-inner product on $\mathcal{H}$ defined by $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$ for all $\xi, \eta \in \mathcal{H}$ then $(\mathcal{H}, \langle\ ,\ \rangle_A)$ is called

* Corresponding author at: Instituto Argentino de Matemática "Alberto P. Calderón", Saavedra 15, Piso 3 (1083), Buenos Aires, Argentina.
E-mail addresses: gfongi@conicet.gov.ar (G. Fongi), celeste.gonzalez@conicet.gov.ar (M.C. Gonzalez).
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a semi-Hilbertian space. In the literature there are many papers that study operators defined on semi-Hilbertian spaces. The first work on this subject is due to Krein [15]. There, the author deals with operators which are A-selfadjoint, i.e., which are selfadjoint respect to the semi-inner product $\langle\ ,\ \rangle_A$. In [10,11] the existence of A-selfadjoint idempotents with a fixed closed range is studied. Later, in [3] operators which are isometric, unitary and partially isometric under the structure induced by $\langle\ ,\ \rangle_A$ are described. One of the main characteristics of the operators defined in $(\mathcal{H}, \langle\ ,\ \rangle_A)$ is that the existence of an adjoint operator for $\langle\ ,\ \rangle_A$ is not guaranteed. Therefore, the extension of certain properties of operators in $L(\mathcal{H})$ to bounded linear operators defined on a semi-Hilbertian space is not trivial.

In this article we continue with the study of partial isometries in the context of semi-Hilbertian spaces that began in [3] and followed in [2]. Recall that $T \in L(\mathcal{H})$ is a partial isometry if $\|T\xi\| = \|\xi\|$ for all $\xi \in N(T)^\perp$, where $N(T)$ denotes the nullspace of T. The following equivalent conditions are well-known for $T \in L(\mathcal{H})$:

1. T is a partial isometry;
2. T^*T is an idempotent operator;
3. TT^* is an idempotent operator;
4. $TT^*T = T$;
5. $T^*TT^* = T^*$;
6. $T^* = T^\dagger$;
7. $T^*\eta$ is the unique least square solution with minimal norm of the equation $T\xi = \eta$ for all $\xi \in \mathcal{H}$;

where T^* and $T^\dagger$ denote the adjoint operator of T and the Moore–Penrose inverse of T, respectively.

Given a positive operator $A \in L(\mathcal{H})$, an operator $T \in L(\mathcal{H})$ is an A-partial isometry if $\|T\xi\|_A = \|\xi\|_A$ for all $\xi \in N(AT)^{\perp_A}$; where $\|\xi\|_A = \langle \xi, \xi\rangle_A^{1/2}$ is the seminorm on $\mathcal{H}$ induced by A and $N(AT)^{\perp_A}$ denotes the orthogonal complement of $N(AT)$ with respect to $\langle\ ,\ \rangle_A$. However this definition does not allow to ensure that an A-partial isometry admits an A-adjoint operator. Therefore, it is not trivial how to extend the equivalences 1 to 7 above to A-partial isometries.

One of the main goals of this article is to analyze whether the equivalences stated above for partial isometries are still valid in the context of semi-Hilbertian spaces. For this purpose we deal with A-partial isometries that admit A-adjoint and we fix a distinguished one. There is a preliminary study in this direction. In [3] an equivalence like $1 \leftrightarrow 2$ above is shown in the context of semi-Hilbertian spaces under different hypotheses, for example the existence of an A-selfadjoint idempotent with a fixed closed range; or the closedness of the range of A. Later, in [2] the relationship between A-partial isometries and generalized inverses is investigated. In particular, the connection between A-partial isometries and A-generalized inverses is described. Here, the A-generalized inverses will play the roll of the Moore–Penrose inverses in the semi-Hilbertian space $(\mathcal{H}, \langle\ ,\ \rangle_A)$. This study is related to equivalences $1 \leftrightarrow 4 \leftrightarrow 6$ above. In Proposition 3.4 we study equiva-