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Robustness analysis of preconditioned successive projection algorithm for general form of separable NMF problem

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ABSTRACT

The successive projection algorithm (SPA) has been known to work well for separable nonnegative matrix factorization (NMF) problems arising in applications, such as topic extraction from documents and endmember detection in hyperspectral images. One of the reasons is in that the algorithm is robust to noise. Gillis and Vavasis showed in [8] that a preconditioner can further enhance its noise robustness. The proof rested on the condition that the dimension d and factorization rank r in the separable NMF problem coincide with each other. However, it may be unrealistic to expect that the condition holds in separable NMF problems appearing in actual applications; in such problems, d is usually greater than r. This paper shows, without the condition $d = r$, that the preconditioned SPA is robust to noise.

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1. Introduction

A d-by-m nonnegative matrix $\boldsymbol{A}$ is said to be *separable* if it has a decomposition of the form

$$\boldsymbol{A} = \boldsymbol{F}\boldsymbol{W} \text{ for } \boldsymbol{F} \in \mathbb{R}_+^{d \times r} \text{ and } \boldsymbol{W} = (\boldsymbol{I}, \boldsymbol{K})\boldsymbol{\Pi} \in \mathbb{R}_+^{r \times m} \tag{1}$$

where $\boldsymbol{I}$ is an r-by-r identity matrix, $\boldsymbol{K}$ is an r-by-$(m-r)$ nonnegative matrix, and $\boldsymbol{\Pi}$ is an m-by-m permutation matrix. Here, we call $\boldsymbol{F}$ the *basis matrix* of $\boldsymbol{A}$ and r the *factorization rank*. The separable nonnegative matrix factorization problem is stated as follows.

(Separable NMF Problem) Let $\boldsymbol{A}$ be of the form given in (1). Find an index set $\mathcal{I}$ with r elements such that $\boldsymbol{A}(\mathcal{I})$ coincides with the basis matrix $\boldsymbol{F}$.

The notation $\boldsymbol{A}(\mathcal{I})$ denotes the submatrix of $\boldsymbol{A}$ whose column indices are in $\mathcal{I}$; in other words, $\boldsymbol{A}(\mathcal{I}) = (\boldsymbol{a}_i : i \in \mathcal{I})$ for the ith column vector $\boldsymbol{a}_i$ of $\boldsymbol{A}$. We use the abbreviation NMF to refer to nonnegative matrix factorization. The problem above can be thought of as a special case of NMF problem. The NMF problem is intractable, and in fact, was shown to be NP-hard in [13]. The authors of [3] proposed to put an assumption, called separability. The separability assumption turns it into a tractable problem referred to as a separable NMF problem. Although the assumption may restrict the range of applications, it is known that separable NMF problems nonetheless can be used for the purpose of topic extraction from documents [4,2,11] and endmember detection in hyperspectral images [7,8].

Several algorithms have been developed for solving the separable NMF problem. One of our concerns is how robust these algorithms are to noise, since it is reasonable to suppose that the separable matrix contains noise in separable NMF problems arising from the applications mentioned above. We consider an algorithm for solving a separable NMF problem and suppose that the separable matrix contains noise. If the algorithm can identify a matrix close to the basis matrix, we say that it is *robust to noise*.

The successive projection algorithm (SPA) was originally proposed in [1] in the context of chemometrics. Currently, the algorithm and its variants are used for topic modeling, document clustering and hyperspectral image unmixing. Gillis and Vavasis showed in [7] that SPA is robust to noise and presented empirical results suggesting that the algorithm is a promising approach to hyperspectral image unmixing. The theoretical results implied that further improvement in noise robustness can be expected if we can make the condition number of the basis matrix smaller. Hence, they proposed in [8] to use a preconditioning matrix for reducing the condition number of the basis matrix. They showed that the noise robustness of SPA is improved by using a preconditioner. The proof rested on the condition that the dimension d and factorization rank r in a separable matrix coincide with each other. However, it may be unrealistic to expect that the