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Mutually orthogonal rectangular gerechte designs

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ABSTRACT

Let q be a prime power and r, s, m, n positive integers. We construct families of mutually orthogonal gerechte designs of order q^{r+s} with rectangular regions of size $q^r \times q^s$. This leads to a lower bound on the size of a family of mutually orthogonal gerechte designs of order mn with rectangular regions of size $m \times n$. The construction is linear-algebraic; surrounding theory employs companion matrices and Toeplitz matrices over finite fields.

1. Introduction

1.1. Purpose and background

Our purpose is to construct gerechte designs possessing rectangular regions, and to investigate mutually orthogonal families of such gerechte designs. We begin with some background and a description of the results in this paper.

Let $m, n \in \mathbb{Z}^+$. An $(\mathbf{m}, \mathbf{n})$**-gerechte design** is a latin square of order mn for which we require that each symbol appears exactly once within each canonical $m \times n$ region. (See Fig. 1.) In this paper, "**rectangular gerechte design**" will mean "(m, n)-gerechte

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0	1	2	3	5	4	7	6
4	5	6	7	1	0	3	2
3	2	1	0	6	7	4	5
7	6	5	4	2	3	0	1
2	3	0	1	7	6	5	4
6	7	4	5	3	2	1	0
1	0	3	2	4	5	6	7
5	4	7	6	0	1	2	3

0	1	2	3	4	5	6	7
4	5	6	7	0	1	2	3
1	0	3	2	5	4	7	6
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
2	3	0	1	6	7	4	5
7	6	5	4	3	2	1	0
3	2	1	0	7	6	5	4

Fig. 1. Two $(2, 4)$-gerechte designs.

00	11	22	33	54	45	76	67
44	55	66	77	10	01	32	23
31	20	13	02	65	74	47	56
75	64	57	46	21	30	03	12
26	37	04	15	72	63	50	41
62	73	40	51	36	27	14	05
17	06	35	24	43	52	61	70
53	42	71	60	07	16	25	34

Fig. 2. The Fig. 1 gerechte designs are orthogonal: upon superimposition, there is no repetition of ordered pairs.

design". We also refer to an (n, n)-gerechte design as a **sudoku solution** of order n^2. For example, solving a newspaper sudoku puzzle results in a sudoku solution with $n = 3$. Also, observe that $(m, 1)$- and $(1, n)$-gerechte designs are merely latin squares of order m and n, respectively. An $(\mathbf{m}, \mathbf{n})$**-gerechte framework** is a template for an (m, n)-gerechte design, with no symbols filled in. Two superimposed (m, n)-gerechte designs form an $mn \times mn$ array of ordered pairs of symbols. When there is no repetition in this array, the two (m, n)-gerechte designs are said to be **orthogonal**. (See Fig. 2.)

Observe that the criteria for orthogonality of two (m, n)-gerechte designs are exactly the criteria for orthogonality of two latin squares. Constructing sets of pairwise mutually orthogonal latin squares (MOLS) is a classical combinatorics problem that has applications to finite geometry, as well as to combinatorial and statistical designs. For $n \geq 2$, one can construct a set of at most $n - 1$ MOLS of order n, and this bound is achieved when n is a prime power. However, it is a longstanding open problem to determine, for any given natural number n, the size of the largest set of MOLS of order n. (For MOLS problems and applications, see, for example, [2,3,12,7,14].)

Since gerechte designs are special cases of latin squares, classical problems concerning latin squares specialize to gerechte designs. In recent years there has been interest in exploring these problems. Regarding orthogonality of sudoku solutions, in [1] it is shown that when n is a prime power one may construct a (largest possible) set of $n(n - 1)$ mutually orthogonal ordinary sudoku solutions of order n^2. This result is also verified in [13] (by means only superficially different from [1]) and it is used, in conjunction with a Kronecker product adapted to sudoku, to produce a lower bound on the maximum size of a set of mutually orthogonal sudoku solutions of order n^2 for *any* natural number n. These bounds are not affected by the addition of certain conditions, such as diagonality ([6] and [9]) and magic subregions [10].