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Distribution of the eigenvalues of a random system of homogeneous polynomials

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ABSTRACT

Let $f = (f_1, \ldots, f_n)$ be a system of n complex homogeneous polynomials in n variables of degree d. We call $\lambda \in \mathbb{C}$ an eigenvalue of f if there exists $v \in \mathbb{C}^n \setminus \{0\}$ with $f(v) = \lambda v$, generalizing the case of eigenvalues of matrices ($d = 1$). We derive the distribution of λ when the f_i are independently chosen at random according to the unitary invariant Weyl distribution and determine the limit distribution for $n \to \infty$.

1. Introduction

The theory of eigenvalues and eigenvectors of matrices is a well-studied subject in mathematics with a wide range of application. However, attempts to generalize this concept to homogeneous polynomial systems of higher degree have only been made very

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recently, motivated by tensor analysis [12,13], spectral hypergraph theory [9] or optimization [10]. An overview on recent publications can be found in [11], where the authors use the term "spectral theory of tensors".

Following Cartwright and Sturmfels, who in [4] adapt Qi's definition of E-eigenvalues, we say that a pair $(v, \lambda) \in (\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}$ is an *eigenpair* of a system $f := (f_1, \dots, f_n)$ of n complex homogeneous polynomials of degree d in the variables $X_1, \dots, X_n$ if $f(v) = \lambda v$. We call v an eigenvector and λ an eigenvalue of f. If in addition $v^T \bar{v} = 1$, we call the pair (v, λ) normalized.

By [7, Theorem 1.3] we expect the task of computing eigenvalues of a given system to be hard. It is therefore natural to ask for the distribution of the eigenvalues, when the system f is random.

In the case $d = 1$ we obtain the definition of eigenpairs of matrices. In [5] Ginibre assumes the entries of a complex matrix $A = (a_{i,j})$ to be independently distributed with density $\pi^{-1} \exp(-|a_{i,j}|^2)$ and describes the distribution of an eigenvalue λ, that is chosen uniformly at random from the n eigenvalues of A.

Can Ginibre's results be extended to arbitrary degree d? The answer is yes and provided in this paper. Let us call two eigenpairs (v, λ), (w, η) *equivalent* if there exists some $t \in \mathbb{C} \setminus \{0\}$, such that $(v, \lambda) = (tw, t^{d-1}\eta)$. Note that if both (v, λ) and $(w, \eta) \in \mathcal{C}$ are normalized, then we must have $|t| = 1$. This implies that the intersection of an equivalence class with the set of normalized eigenpairs of f is a circle, that we assume to have volume 2π. Cartwright and Sturmfels point out in [4, Theorem 1] that if $d > 1$, the number of equivalence classes of eigenpairs of a generic f is $D(n, d) := (d^n - 1)/(d - 1)$.

We define a probability distribution on the space of eigenvalues as follows:

1. For each $1 \leq i \leq n$ choose f_i independently at random with the density $\pi^{-k} \exp(-\|f_i\|^2)$, where $k := \binom{n-1+d}{d}$. Here $\| \, \|$ is the unitary invariant norm on the space of homogeneous polynomials of degree d defined in Section 3, see also [3, sec. 16.1]. (The resulting distribution of the f_i is sometimes called the Weyl distribution.)
2. Among the $D(n, d)$ many equivalence classes $\mathcal{C}$ of eigenpairs of f, choose one uniformly at random.
3. Choose an normalized eigenpair $(v, \lambda) \in \mathcal{C}$ uniformly at random.
4. Apply the projection $(v, \lambda) \mapsto \lambda$.

We denote by $\rho^{n,d} : \mathbb{C} \to \mathbb{R}_{\geq 0}$, $\lambda \mapsto \rho^{n,d}(\lambda)$ the density of the resulting probability distribution. Observe that if $d = 1$, then $\rho^{n,1}$ is the density of Ginibre's distribution.

The unitary invariance of $\| \, \|^2$ implies that $\rho^{n,d}(\lambda)$ only depends on $|\lambda|$, but not on the argument of λ. We therefore introduce the following notation:

$$R := 2\,|\lambda|^2 . \tag{1.1}$$