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Notes on Birkhoff–von Neumann decomposition of doubly stochastic matrices

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ABSTRACT

Birkhoff–von Neumann (BvN) decomposition of doubly stochastic matrices expresses a double stochastic matrix as a convex combination of a number of permutation matrices. There are known upper and lower bounds for the number of permutation matrices that take part in the BvN decomposition of a given doubly stochastic matrix. We investigate the problem of computing a decomposition with the minimum number of permutation matrices and show that the associated decision problem is strongly NP-complete. We propose a heuristic and investigate it theoretically and experimentally on a set of real world sparse matrices and random matrices.

1. Introduction

Let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$. The matrix $\mathbf{A}$ is said to be doubly stochastic if $a_{ij} \geq 0$ for all i, j and $\mathbf{A}e = \mathbf{A}^T e = e$, where e is the vector of all ones. By Birkhoff's Theorem, there exist $\alpha_1, \alpha_2, \ldots, \alpha_k \in (0, 1)$ with $\sum_{i=1}^{k} \alpha_i = 1$ and permutation matrices $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_k$

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such that:

$$\mathbf{A} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2 + \cdots + \alpha_k \mathbf{P}_k \,. \tag{1}$$

This representation is also called Birkhoff–von Neumann (BvN) decomposition. We refer to the scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ as the coefficients of the decomposition. Such a representation of $\mathbf{A}$ as a convex combination of permutation matrices is not unique in general. For any such representation the *Marcus–Ree Theorem* [1] states that $k \leq n^2 - 2n + 2$ for dense matrices; Brualdi and Gibson [2] and Brualdi [3] show that for a fully indecomposable sparse matrix with τ nonzeros $k \leq \tau - 2n + 2$.

We are interested in the problem of finding the minimum number k of permutation matrices in the representation (1). More formally we investigate the MinBvNDec problem defined as follows:

Input: A doubly stochastic matrix $\mathbf{A}$.
Output: A Birkhoff–von Neumann decomposition of $\mathbf{A}$ as

$$\mathbf{A} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2 + \cdots + \alpha_k \mathbf{P}_k.$$

Measure: The number k of permutation matrices in the decomposition.

Brualdi [3, p. 197] investigates the same problem and concludes that this is a difficult problem. We continue along this line and show that the MinBvNDec problem is NP-hard (Section 2). We also propose a heuristic (Section 3) for obtaining a BvN decomposition with a small number of permutation matrices. We investigate some of the properties of the heuristic theoretically and experimentally (Section 4).

2. The minimum number of permutation matrices

We show in this section that the decision version of the problem is NP-complete. We first give some definitions and preliminary results.

2.1. Preliminaries

A multi-set can contain duplicate members. Two multi-sets are equivalent if they have the same set of members with the same number of repetitions.

Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices. We write $\mathbf{A} \subseteq \mathbf{B}$ to denote that for each nonzero entry of $\mathbf{A}$, the corresponding entry of $\mathbf{B}$ is nonzero. In particular, if $\mathbf{P}$ is an $n \times n$ permutation matrix and $\mathbf{A}$ is a nonnegative $n \times n$ matrix, $\mathbf{P} \subseteq \mathbf{A}$ denotes that the entries of $\mathbf{A}$ at the positions corresponding to the nonzero entries of $\mathbf{P}$ are positive. We use $\mathbf{P} \odot \mathbf{A}$ to denote the entrywise product of $\mathbf{P}$ and $\mathbf{A}$, which selects the entries of $\mathbf{A}$ at the positions corresponding to the nonzero entries of $\mathbf{P}$. We also use $\min\{\mathbf{P} \odot \mathbf{A}\}$ to denote the minimum entry of $\mathbf{A}$ at the nonzero positions of $\mathbf{P}$.