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A sectional curvature for statistical structures ☆

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A new type of sectional curvature is introduced. The notion is purely algebraic and can be located in linear algebra as well as in differential geometry.

1. Introduction

Sectional curvature is one of the most important concepts in differential geometry. Nevertheless, it is attributed to Riemannian or pseudo-Riemannian geometry only. The curvature tensor field is defined for any connection but to define a sectional curvature, which assigns to a vector plane of a tangent space a number, seems to need a metric tensor. Moreover, the metric and the connection must be related in a good manner. For

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instance, in the classical affine differential geometry one has a metric tensor field and the so called induced connection, but the curvature tensor of type $(0, 4)$ constructed by these objects does not have enough symmetries. The tensor satisfies appropriate symmetry conditions for affine spheres but it leads to trivial cases, namely to spaces of constant sectional curvature. The problem can be solved by taking the average of the curvature tensor and the curvature tensor for the dual connection. This idea is discussed in [8] for statistical structures on abstract manifolds, that is, on manifolds (not necessarily immersed in any standard space) endowed with a metric tensor field g and a torsion-free affine connection ∇ for which ∇g as a 3-covariant tensor is symmetric.

A statistical structure is also called a Codazzi structure, see e.g. [7,6]. We use the name "statistical structure" following [5] or [3]. The name "Codazzi structure" may refer to all situations, where we have any tensor field whose covariant derivative is totally symmetric. Many useful formulas for Codazzi structures can be found in [10].

The geometry of affine hypersurfaces in the standard affine space $\mathbf{R}^n$ or, more generally, the geometries of the second fundamental form, including the theory of Lagrangian submanifolds in complex space forms, are natural sources of statistical structures. However, the fact that the structures are induced by the simple structures on the ambient spaces imposes strong conditions on the induced statistical structure. For instance, for affine hypersurfaces, it is necessary that the dual connection is projectively flat.

It turns out that for statistical structures one can define few sectional curvatures. In [8] we studied the sectional ∇-curvature, that is, a sectional curvature determined by a metric tensor and a connection ∇. In this paper we propose another type of sectional curvature. Its idea is purely algebraic. This sectional curvature can be defined on any vector space endowed with a scalar product and a symmetric cubic form. Then it can be transfered to statistical structures on manifolds. In this paper we provide some basic information on this sectional curvature and we give few examples of theorems concerning this notion. It turns out that the K-sectional curvature influences a statistical structure in a strong way. For instance, some assumptions on the curvature imply that the underlying metric structure must be flat. The fact that the sectional K-curvature is constant determines the difference tensor K point-wisely and in some cases also locally.

2. Statistical structures

One can define a statistical structure on a manifold M in three equivalent ways. First of all M must have a Riemannian structure defined by a metric tensor field g. Throughout the paper we assume that g is positive definite, although g can be also indefinite. A statistical structure can be defined as a pair (g, K) on a manifold M, where g is a Riemannian metric tensor field and K is a symmetric $(1, 2)$-tensor field which is also symmetric relative to g, that is, the cubic form

$$C(X, Y, Z) = g(X, K(Y, Z)) \tag{1}$$