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Angles, triangle inequalities, correlation matrices and metric-preserving and subadditive functions

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ABSTRACT

We present inequalities concerning the entries of correlation matrices, density matrices, and partial isometries through the positivity of 3×3 matrices. We extend our discussions to the inequalities concerning the triangle triplets with metric-preserving and subadditive functions.

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1. Introduction

We begin our discussions with angles between vectors by looking at two angle definitions and their characteristics. We first study the triangle inequalities for the angles through the positivity (i.e., positive semidefiniteness) of a 3×3 matrix; we investigate the relations between the triangle inequalities (of angles or more generally the triangle triplets) and metric-preserving and subadditive functions. Our results will capture some existing ones but using a different approach. The discussion on the 3×3 positive semidefinite matrices leads to some inequalities concerning the entries of correlation matrices with which we obtain inequalities for density matrices and partial isometries.

Let V be an inner product space with the inner product $\langle \cdot, \cdot \rangle$ over the real number field $\mathbb{R}$. For any two nonzero vectors u, v in V, there are two common ways to define the *angle* between the vectors u and v in terms of the inner product (see, for instance, [10, p. 58] and [11, p. 335], respectively):

$$\theta(u, v) = \arccos \frac{|\langle u, v\rangle|}{\|u\|\|v\|} \tag{1}$$

$$\Theta(u, v) = \arccos \frac{\langle u, v\rangle}{\|u\|\|v\|} \tag{2}$$

There are various reasons that the angles are defined in ways (1) and (2) (in the sense of Euclidean geometry). Definition (1) may stem from the angles between subspaces, while (2) makes perfect sense intuitively. We are interested in the properties of the angles regardless of their definitions.

Angle and inner product can be viewed as counterparts in a vector space. A vector may take a simple and familiar form like the ones in $\mathbb{R}^2$; it may look much more complicated like the elements (linear combinations of wedge products) in the Grassmann spaces [14, p. 172]. Some matrix functions are closely related to vectors and some types of products of vectors. It is a well-known fact that the trace of a matrix product is an inner product: $\operatorname{tr}(AB) = \langle B, A^* \rangle$. The determinant (even more generally, the generalized matrix functions) can be expressed as an inner product of $*$-tensors (see, e.g., [14, p. 226]).

We will also need the term *correlation matrix*, which is a positive semidefinite matrix with all main diagonal entries equal to 1. Every positive semidefinite matrix with nonzero main diagonal entries can be normalized to a correlation matrix through scaling. The correlation matrices are frequently used in statistics. For its determinant and permanent properties, see, e.g., [16,20,23]. Our theorems rely on the results for the 3×3 correlation matrices.

In Section 2, we focus on the triangle inequalities through the positivity of 3×3 matrices. Our results provide a unified proof for the triangle inequalities for the angles θ and Θ. We also present some relationships between the elements of correlation matrices. As applications, we obtain inequalities for density matrices and partial isometries. In