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Squaring operator Pólya–Szegö and Diaz–Metcalf type inequalities

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We square operator Pólya–Szegö and Diaz–Metcalf type inequalities as follows: If operator inequalities $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$ hold for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, then for every unital positive linear map Φ the following inequalities hold:

$$(\Phi(A)\sharp\Phi(B))^2 \leq \left(\frac{M_1M_2 + m_1m_2}{2\sqrt{M_1M_2m_1m_2}}\right)^4 \Phi(A\sharp B)^2$$

and

$$\left(\frac{M_2m_2}{M_1m_1}\Phi(A) + \Phi(B)\right)^2 \leq \left(\frac{(M_1m_1(M_2^2 + m_2^2) + M_2m_2(M_1^2 + m_1^2))^2}{8\sqrt{M_2M_1m_1m_2}M_1^2m_1^2M_2m_2}\right)^2 \Phi(A\sharp B)^2.$$

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1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. Throughout the paper, a capital letter means an operator in $\mathbb{B}(\mathscr{H})$. If $\dim \mathscr{H} = n$, then $\mathbb{B}(\mathscr{H})$ can be identified with the space $\mathbb{M}_n$ of all $n \times n$ complex matrices. We identify a scalar with the identity operator I multiplied by this scalar. An operator A is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$, and we then write $A \geq 0$. An operator A is said to be strictly positive (denoted by $A > 0$) if it is a positive invertible operator. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $B \geq A$ if $B - A \geq 0$. For strictly positive operators, $A^2 \leq k^2 B^2$ for some constant k if and only if $(AB^{-1})^*(AB^{-1}) \leq k^2$ and this occurs if and only if $\|AB^{-1}\| \leq k$. A linear map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ is called positive if $A \geq 0$ implies $\Phi(A) \geq 0$. If this implication holds for $>$ instead of $\geq$, we say that Φ is strictly positive. It is said to be unital if Φ preserves the identity operator. The operator norm is denoted by $\|\cdot\|$. For $A, B > 0$, the operator geometric mean $A\sharp B$ is defined by $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$. Using a standard limit argument, this notion can be extended for positive operators A, B. The geometric mean operation is monotone, in the sense that $C_1 \leq D_1$ and $C_2 \leq D_2$ imply that $C_1\sharp C_2 \leq D_1\sharp D_2$.

Moslehian et al. [12, Theorem 2.1] gave operator Pólya–Szegö inequality (see also [8] for an interesting proof for matrices) and Diaz–Metcalf type inequality as follows:

Theorem 1.1. *Let Φ be a positive linear map. If $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, then*

$$\Phi(A)\sharp\Phi(B) \leq \alpha \cdot \Phi(A\sharp B), \tag{1.1}$$

where

$$\alpha := \frac{1}{2}\left\{\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right\}.$$

Theorem 1.2. *Let Φ be a positive linear map. If $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$ and $0 < m_2 \leq M_2$, then the following inequality holds:*

$$\frac{M_2m_2}{M_1m_1}\Phi(A) + \Phi(B) \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right)\Phi(A\sharp B). \tag{1.2}$$

It is well known that t^s for any $0 \leq s \leq 1$ is operator monotone but not so is t^2; see [13]. However, Fujii et al. [5, Theorem 6] used the Kantorovich inequality to show that t^2 is order preserving in a certain sense as follows:

Theorem 1.3. *Let $0 < m \leq A \leq M$ and $A \leq B$. Then*

$$A^2 \leq \frac{(M+m)^2}{4Mm}B^2.$$