Linear Algebra and its Applications 491 (2016) 101–122

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the complexity of the positive semidefinite zero forcing number

Shaun Fallat [a,*,1], Karen Meagher [a,2], Boting Yang [b,3]

[a] *Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada*
[b] *Department of Computer Science, University of Regina, Regina, Saskatchewan S4S 0A2, Canada*

A R T I C L E I N F O

Article history:
Received 21 July 2014
Accepted 10 March 2015
Available online 18 March 2015
Submitted by F. Zhang

MSC:
05C35
05C50
05C78
05C85
68R10

Keywords:
Positive zero forcing number
Clique cover number
Chordal graphs
Computational complexity

A B S T R A C T

The positive semidefinite zero forcing number of a graph is a graph parameter that arises from a non-traditional type of graph colouring and is related to a more conventional version of zero forcing. We establish a relation between the zero forcing and the fast–mixed searching, which implies some NP-completeness results for the zero forcing problem. Relationships between positive semidefinite zero forcing sets and clique coverings are well-understood for chordal graphs. Building upon constructions associated with optimal tree covers and forest covers, we present a linear time algorithm for computing the positive semidefinite zero forcing number of chordal graphs. We also prove that it is NP-complete to determine whether a graph has a positive semidefinite zero forcing set with an additional property.

* Corresponding author.
E-mail addresses: Shaun.Fallat@uregina.ca (S. Fallat), Karen.Meagher@uregina.ca (K. Meagher), Boting.Yang@uregina.ca (B. Yang).

[1] Research supported in part by an NSERC Discovery Research Grant, Application No.: RGPIN-2014-06036.
[2] Research supported in part by an NSERC Discovery Research Grant, Application No.: RGPIN-341214-2013.
[3] Research supported in part by an NSERC Discovery Research Grant, Application No.: RGPIN-2013-261290.

http://dx.doi.org/10.1016/j.laa.2015.03.011
0024-3795/

1. Introduction

The zero forcing number of a graph is an important graph parameter that was introduced in [2]. In [4], the concept of the *positive semidefinite* zero forcing number, or, simply, *positive* zero forcing number, was developed. First and foremost, the interest in these parameters has been to provide a bound on the maximum nullities (or, equivalently, the minimum rank) of certain symmetric matrices associated with graphs. Recently, this parameter has been considered in other contexts, see, for example, [3,20]. Physicists have also studied this parameter in conjunction with control of quantum systems; in this context it is known as the *graph infection number* [8,19].

The same notion arises in computer science within the context of graph searching. At its most basic level, edge search and node search models represent two significant graph search problems [15,16]. Bienstock and Seymour [5] introduced the mixed search problem that combines the edge search and the node search problems. Dyer et al. [9] introduced the fast search problem. Recently, a fast–mixed search model was introduced in an attempt to combine fast search and mixed search models [21]. For this model, we assume that the simple graph G contains a single *fugitive*, invisible to the searchers, that can move at any speed along a "searcher-free" path and hides on vertices or along edges. In this case, the minimum number of searchers required to capture the fugitive is called the *fast–mixed search number of* G. As we will see, the fast–mixed search number and the zero forcing number of G are indeed equal.

Suppose that G is a simple finite graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We begin by specifying a set of initial vertices of the graph (which we say are coloured black, while all other vertices are white). Then, using a designated colour change rule applied to these vertices, we progressively change the colour of white vertices in the graph to black. Our colouring consists of only two colours (black and white) and the objective is to colour all vertices black by repeated application of the colour change rule to our initial set. In general, we want to determine the smallest set of vertices that must be black initially, to eventually change all of the vertices in the graph to black.

The conventional zero forcing rule results in a partition of the vertices of the graph into sets, such that each such set induces a path in G. Further, each of the initial black vertices is an end point of one of these paths. More recently a refinement of the colour change rule, called the positive zero forcing colour change rule, was introduced. Using this rule, the positive zero forcing number was defined (see, for example, [4,10,11]). When the positive zero forcing colour change rule is applied to a set of initial vertices of a graph, the vertices are then partitioned into sets, so that each such set induces a tree in G.

As mentioned above, one of the original motivations for studying these parameters is that they both provide an upper bound on the maximum nullity of both symmetric and positive semidefinite matrices associated with a graph (see [3,4]). All matrices considered in this paper have real entries. For a given graph $G = (V, E)$, define

$$\mathcal{S}(G) = \{A = [a_{ij}] : A = A^T,\ \text{for } i \neq j, a_{ij} \neq 0 \text{ if and only if } \{i, j\} \in E(G)\}$$