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A combinatorial equivalence relation for formal power series

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In this paper we examine an equivalence relation on the set of formal power series with nonzero constant term. This is done both in terms of functional equations and also by interlacing two concepts from Riordan group theory, the A-sequence and the Bell subgroup. The best known example gives an equivalence class

$$\{1+z, 1/(1-z), C(z), T(z), Q(z), \ldots\}$$

where $C(z), T(z)$ and $Q(z)$ are generating functions of the Catalan numbers, ternary numbers and quaternary numbers, respectively. A power series for one member of an equivalence class can be transformed into power series for the rest of members in the equivalence class and interpretations in terms of weighted lattice paths can also be given.

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1. Introduction

Let $\mathbb{R}[[z]]$ be the ring of formal power series over the real field $\mathbb{R}$ and let

$$\mathfrak{F}_k = \left\{ G(z) = \sum_{n=k}^{\infty} g_n z^n \in \mathbb{R}[[z]] \mid g_k \neq 0 \right\}.$$

We will usually shorten $G(z)$ to the less cumbersome G.

The Riordan group [17] is the set of infinite lower triangular matrices called *Riordan matrices* $\mathcal{M} = [m_{ij}]_{i,j\in\mathbb{N}_0}$ defined by $m_{ij} = [z^i]GF^j$ for a pair $G \in \mathfrak{F}_0$ and $F \in \mathfrak{F}_1$ where $[z^i]$ is the coefficient extraction operator and $\mathbb{N}_0 = \{0, 1, \ldots\}$. Usually we write $\mathcal{M} = (G, F)$. In particular, a Riordan matrix of the form (G, zG) is called a Bell matrix and the Bell matrices form a subgroup of the Riordan group. The multiplication for the Riordan group is just matrix multiplication which can be described in terms of generating functions as follows:

$$(G, F)\,(H, L) = (GH(F), L(F))\,.$$

It is easy to see that $(1, z)$ is the identity matrix and $(1/G(F^{(-1)}), F^{(-1)})$ is the inverse matrix of (G, F) where $F^{(-1)}$ is the compositional inverse of F i.e., $F(F^{(-1)}) = F^{(-1)}(F) = z$. The existence of inverses in the Riordan group relies on the fact that a formal power series F has a multiplicative inverse if and only if $F \in \mathfrak{F}_0$ and a compositional inverse if and only if $F \in \mathfrak{F}_1$.

It is known [14,16] that every Riordan matrix $\mathcal{M} = (G, F)$ has a unique horizontal sequence called the A-*sequence* $(a_k)_{k\in\mathbb{N}_0}$ such that $m_{i+1,j+1} = \sum_{k\geq 0} a_k m_{i,j+k}$. If A is the generating function for the A-sequence then we have the equation $F = zA(F)$. The best known example is for the Pascal triangle matrix where $A(z) = 1+1\cdot z+0\cdot z^2+0\cdot z^3+\cdots = 1 + z$.

In this paper, we are interested in an equivalence relation on the set $\mathfrak{F}_0$ using the A-sequence of a Riordan matrix. The basic idea is to start with a function $G := G_0 \in \mathfrak{F}_0$, use it as the A-sequence generating function to form a Bell matrix (G_1, zG_1) and then to use G_1 as the A-sequence to form (G_2, zG_2). Repeating this process leads to G_3, G_4, and G_n for all positive integers n. Since the process is reversible we also can find G_{-n}. By repeated use of $F = zA(F)$ we find that $G_n = G(zG_n^n)$ for all $n \in \mathbb{Z}$. In Section 2, we show that this process yields a natural equivalence relation on $\mathfrak{F}_0$ where the set $\{G_n | n \in \mathbb{Z}\}$ is the equivalence class for G. As an application, starting with a quadratic functional equation we can move on to finding power series solutions for a large class of functional equations of a higher-degree. This concept will be described in Section 3. Finally, in Section 4 we give a unified combinatorial interpretation for every element of the equivalence class of G that satisfies the functional equation $G = 1 + \sum_{i,j\geq 1} \omega_{i,j} z^i G^j$.