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Essential spectral equivalence via multiple step preconditioning and applications to ill conditioned Toeplitz matrices

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A B S T R A C T

In this note, we study the fast solution of Toeplitz linear systems with coefficient matrix $T_n(f)$, where the generating function f is nonnegative and has a unique zero at zero of any real positive order θ. As preconditioner we choose a matrix $\tau_n(f)$ belonging to the so-called τ algebra, which is diagonalized by the sine transform associated to the discrete Laplacian. In previous works, the spectral equivalence of the matrix sequences $\{\tau_n(f)\}_n$ and $\{T_n(f)\}_n$ was proven under the assumption that the order of the zero is equal to 2: in other words the preconditioned matrix sequence $\{\tau_n^{-1}(f)T_n(f)\}_n$ has eigenvalues, which are uniformly away from zero and from infinity. Here we prove a partial generalization of the above result when $\theta < 2$. Furthermore, by making use

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Spectral analysis
PCG method

of multiple step preconditioning, we show that the matrix sequences $\{\tau_n(f)\}_n$ and $\{T_n(f)\}_n$ are essentially spectrally equivalent for every $\theta > 2$, i.e., for every $\theta > 2$, there exist m_θ and a positive interval $[\alpha_\theta, \beta_\theta]$ such that all the eigenvalues of $\{\tau_n^{-1}(f)T_n(f)\}_n$ belong to this interval, except at most m_θ outliers larger than β_θ: while the essential bound from above is proven, the bound from below is only observed numerically. Such a nice property, already known only when θ is an even positive integer greater than 2, is coupled with the fact that the preconditioned sequence has an eigenvalue cluster at one, so that the convergence rate of the associated preconditioned conjugate gradient method is optimal. As a conclusion we discuss possible generalizations and we present selected numerical experiments.

1. Introduction

Our goal is to design and analyze a preconditioning technique for the fast solution of a Toeplitz system with $n \times n$ coefficient matrix $T_n(f)$, where f is a given function having a unique zero at zero of positive order θ: the entry (j,k), $1 \leq j,k \leq n$, of the matrix $T_n(f)$ is the l-th Fourier coefficient of f with $l = j - k$ and

$$a_l = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-\mathbf{i}lt}\, dt.$$

The preconditioner is chosen in the so-called τ algebra which is the set of all real symmetric matrices diagonalized by the sine transform associated to the discrete Laplacian (see (2.3)): the preconditioner is chosen to have as eigenvalues a uniform sampling of the symbol f and is denoted by $\tau_n(f)$.

We study the spectrum of the matrix sequences $\{\mathcal{A}_n\}_n$ with $\mathcal{A}_n = \tau_n^{-1}(f)T_n(f)$ and with the goal of localizing the eigenvalues and understanding the asymptotic behavior. We recall that the study of such a matrix sequence gives precise information on the convergence speed of the related Preconditioned Conjugate Gradient (PCG) method. We remark that if we explicitly use the proposed preconditioner in the PCG method, then the total arithmetic cost remains optimal i.e. $O(n \log(n))$ ops in a sequential machine and $O(\log(n))$ in PRAM model with n processors. The same observations are valid also in a multidimensional case, if the order of the zeros is at most two (see the negative results presented in [19] for higher order zeros). In addition to that, the associated preconditioning strategy can be used also in connection with multigrid schemes: as a noteworthy application, see [12,11] for the use of fast Toeplitz preconditioning in the context of a multigrid method for a Collocation/Galerkin isogeometric analysis approximation [6] to the solution of elliptic partial differential equations. In that setting, it is shown that neither a standard multigrid alone nor a preconditioned Krylov solver is both optimal