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Riordan arrays, generalized Narayana triangles, and series reversion

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ABSTRACT

Using elements of the group of Riordan arrays we define a family of generalized Narayana triangles and their associated generalized Catalan numbers, and study their links to series reversion. In particular we use Lagrange inversion techniques to determine the generating functions for these generalized Catalan numbers.

1. Introduction

In this note, we shall use the Riordan group of matrices [11] to find generating functions and identities for objects of combinatorial interest. For instance, we shall find generating functions and identities of the form

$$\binom{2n+1}{n} = (n+1)[x^{n+1}]\ln(c(x)),$$

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where $c(x)$ is the g.f. of the Catalan numbers given by

$$c(x)=\frac{1-\sqrt{1-4x}}{2x},$$

$$F_{n+2}=\sum_{j=0}^{n}\frac{j+1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}(-1)^{n-j-i}\binom{2n-2j-i-1}{n-j-i}\frac{i+0^{n-j+i}}{n-j+0^{(n-j)i}},$$

and

$$\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\sum_{j=0}^{n}\binom{j}{n-j}\binom{j}{k}$$
$$=[x^{n+1}]\frac{1}{\sqrt{5}}\ln\left(\frac{1+3\sqrt{5}+8\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{27x+11}{16}\right)\right)}{1-3\sqrt{5}+8\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{27x+11}{16}\right)\right)}\right).$$

A less elementary example of our results is given by

$$\sum_{k=0}^{n}\binom{n+1}{k}\binom{\frac{n+k}{2}}{k}\frac{1+(-1)^{n-k}}{2}=[x^n]\frac{d}{dx}\ln\sqrt{\frac{f(x)+1}{f(x)-1}},$$

where

$$f(x)=\frac{x}{3}-\frac{2\sqrt{x^2-3x+3}}{3}\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{x(2x^2-9x-18)}{2(x^2-3x+3)^{\frac{3}{2}}}\right)\right).$$

The methods that we shall use will be based on Lagrangian inversion [9] and Riordan arrays [11]. A major role will be played by generalizations of the Narayana numbers. The power of Riordan arrays to produce combinatorial identities has long been recognized, and indeed this was one of the motivating factors in their development. The group structure of the set of Riordan arrays often plays a role in this, in that the existence of an easily described inverse leads to many of these identities. One tool that is well adapted to finding inverse expressions for Riordan arrays is Lagrange inversion, and in particular, its Lagrange–Bürmann incarnation.

We shall use this version of Lagrange inversion to find expressions for the following "Narayana" type expressions

$$\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}T_{n,k}$$

where in most cases $T_{n,k}$ will be the general term of a suitable Riordan array.

Note that in the above we have used the notation "0^n" to signify the sequence with generating function 1. In other words, we adopt the convention that $0^0=1$ and $0^n=0$ for $n\in\mathbb{N}$, $n>0$.