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The minimum rank problem for circulants

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ABSTRACT

The *minimum rank problem* is to determine for a graph G the smallest rank of a Hermitian (or real symmetric) matrix whose off-diagonal zero-nonzero pattern is that of the adjacency matrix of G. Here G is taken to be a circulant graph, and only circulant matrices are considered. The resulting graph parameter is termed the *minimum circulant rank* of the graph. This value is determined for every circulant graph in which a vertex neighborhood forms a consecutive set, and in this case is shown to coincide with the usual minimum rank. Under the additional restriction to positive semidefinite matrices, the resulting parameter is shown to be equal to the smallest number of dimensions in which the graph has an orthogonal representation with a certain symmetry property, and also to the smallest number of terms appearing among a certain family of polynomials determined by the graph. This value is then determined when the number of vertices is prime. The analogous parameter over $\mathbb{R}$ is also investigated.

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1. Introduction

The location of the off-diagonal nonzero entries of a Hermitian or real symmetric matrix can naturally be specified by a graph. More formally, we have the following.

Definition 1.1. Let A be an $n \times n$ Hermitian matrix and G be a simple graph on n vertices, say with $V(G) = \{v_1, v_2, \ldots, v_n\}$. We say that G is the *graph* of A if it is the case that $\{v_i, v_j\} \in E(G)$ if and only if $a_{ij} \neq 0$, for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.

A problem of interest in combinatorial matrix theory is to determine particular ways in which the graph of a matrix constrains its rank. Because the diagonal entries of the matrix play no role in Definition 1.1, every graph allows a diagonally dominant matrix, so the question of how large the rank may be is not interesting. On the other hand, to determine the smallest rank among all matrices with a given graph is an interesting problem, known as the *minimum rank problem* for graphs. More formally, the problem is to determine the value of the graph parameter defined as follows.

Notation 1.2. Let G be a graph. We write $\mathcal{H}(G)$ for the set of all complex Hermitian matrices with graph G.

Definition 1.3. Let G be a graph. The *minimum rank* of G is

$$\operatorname{mr}(G) = \min\{\operatorname{rank}(A) : A \in \mathcal{H}(G)\}.$$

The present work focuses on the case in which G is a circulant graph. We may then consider the smallest rank among all Hermitian (or real symmetric) circulant matrices whose off-diagonal nonzero entries occur according to the edges of G.

A question that naturally arises is: When G is a circulant *graph*, under what conditions is the smallest rank among all Hermitian (or real symmetric) matrices with graph G attained by a circulant *matrix?* In Section 5 we show that this in fact does occur for at least one broad class of circulant graphs, namely those in which each vertex neighborhood comprises a consecutive set of vertices. We also give examples of circulants for which this does not occur, however the problem of providing a complete characterization of such circulants remains open.

We also investigate the problem in the positive semidefinite setting. First, in Section 3, we show that the problem of determining the smallest rank among all positive semidefinite circulant matrices with a given graph is equivalent to determining the smallest number of dimensions admitting an orthogonal representation for the graph with a specific symmetry property. Then, in Section 4, this problem in turn is shown to be equivalent to determining the smallest number of terms in a real polynomial with non-negative coefficients whose zeros intersect a precise subset, determined by the graph, of the complex roots of unity. In Section 5, this value is determined for two broad classes of circulants. Finally, Section 6 develops analogous results over $\mathbb{R}$.