Linear Algebra and its Applications 491 (2016) 419–433

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Green operators of networks with a new vertex

A. Carmona, A.M. Encinas, S. Gago *, M. Mitjana

Departament de Matemàtiques, UPC, Spain

ARTICLE INFO

Article history:
Received 31 January 2015
Accepted 10 November 2015
Available online 19 November 2015
Submitted by R. Brualdi

MSC:
15A09
31C20
05C50

Keywords:
Discrete elliptic operators
Green function
Perturbations
Networks

ABSTRACT

Any elliptic operator defines an automorphism on the orthogonal subspace to the eigenfunctions associated with the lowest eigenvalue, whose inverse is the orthogonal Green operator. In this study, we show that elliptic Schrödinger operators on networks that have been obtained by adding a new vertex to a given network, can be seen as perturbations of the Schrödinger operators on the initial network. Therefore, the Green function of the new network can be computed in terms of the Green function of the original network.

1. Introduction

Discrete elliptic operators can be seen as the discrete counter part of elliptic partial differential operators. In particular, positive semi–definite Schrödinger operators defined on a finite network are examples of those self–adjoint operators. Any elliptic operator defines an automorphism on the orthogonal subspace to the eigenfunctions associated with the lowest eigenvalue, whose inverse is the orthogonal Green operator.

* Corresponding author.
E-mail address: silvia.gago@upc.edu (S. Gago).

http://dx.doi.org/10.1016/j.laa.2015.11.012

In [4], some of the authors analyzed the effect of a perturbation of the network by computing the effective resistance of the perturbed networks through Sherman–Morrison–Woodbury like–formulas, instead of using the Sherman–Morrison formula recursively. In fact, since adding edges to a network does not modify the space of functions on the vertex set of the network, this class of perturbation was placed into the general framework of perturbations of discrete elliptic operators. Specifically, we showed that this problem corresponds with the superposition of rank one perturbations that are orthogonal to the eigenfunction associated with the lowest eigenvalue of the elliptic operator.

The scenario changes when the perturbation consists on adding new vertices to the network. Only few works have tackled the problem of adding a new vertex, see for instance [6,8]. In this work, we consider perturbations that consist on adding a new vertex to a network. After some well–known operations on the Schrödinger operator of the perturbed network, that involves the inverse of the Schur complement of the block corresponding to the added vertices, we show that this Schur complement can be seen as a perturbation of the Schrödinger operator of the original network, understood as a discrete elliptic operator, that is a superposition of rank one perturbations that, this time, are not orthogonal to the eigenfunction associated with the lowest eigenvalue of the elliptic operator. Therefore, we can apply the general theory developed in [4] for this kind of perturbations.

We start the study by revisiting the perturbation of an elliptic operator with a sum of projections that can be, or not, orthogonal to the eigenfunction associated with the smallest eigenvalue. Thus, we consider the relation between the Green operator of the new operator in terms of the Green operator of the previous one.

Next section is devoted to the application of the mentioned results to the addition of a new vertex to the network Γ in order to get a network Γ'. Moreover, we obtain the relation between the Schrödinger operators of the two networks Γ and Γ' and in addition, we give the explicit expression of the matrix associated with the Green operator. We finally verify the results we have obtained. More specifically, we compute the Green function of a non–complete wheel when we interpret this network as the addition of a new vertex to a cycle, in an appropriate way.

2. Specific notation and preliminary results

Given a finite set V of n elements, we denote by $\mathcal{C}(V)$ the space of real valued functions on V. For any vertex $x \in V$, the Dirac function at x is denoted by $\varepsilon_x \in \mathcal{C}(V)$; the scalar product on $\mathcal{C}(V)$ is $\langle u, v\rangle = \sum_{x\in V} u_x v_x$ for each $u, v \in \mathcal{C}(V)$. A unitary and positive function ω is called a weight and $\Omega(V)$ denote the set of weights.

If $\mathcal{K}$ is an endomorphism of $\mathcal{C}(V)$, it is self–adjoint when $\langle \mathcal{K}(u), v\rangle = \langle u, \mathcal{K}(v)\rangle$ for any $u \in \mathcal{C}(V)$. Moreover, $\mathcal{K}$ is positive semi–definite when $\langle \mathcal{K}(u), u\rangle \geq 0$ for any $u \in \mathcal{C}(V)$. A self–adjoint operator $\mathcal{K}$ is elliptic if it is positive semi–definite and its lowest eigenvalue λ is simple. Moreover, there exists a unique unitary function $\omega \in \mathcal{C}(V)$, up to sign, satisfying $\mathcal{K}(\omega) = \lambda\omega$, so $\mathcal{K}$ is called (λ, ω)–elliptic operator. It is straightforward that