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Continuous maps preserving local spectra of matrices

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ABSTRACT

We characterize continuous maps on matrix spaces which compress/expand the local spectrum at each vector. We also characterize continuous maps on matrices which decrease/increase at each vector some expressions defined in terms of the local spectral radius and the local inner spectral radius.

1. Introduction and preliminaries

The study of both linear and nonlinear local spectra preserver problems attracted the attention of several authors over the last decade. A. Bourhim and T. Ransford were the first ones to consider this type of preserver problem, characterizing in [4] additive maps on $\mathcal{L}(X)$, the algebra of all linear bounded operators on a complex Banach space X, which preserve the local spectrum of each $T \in \mathcal{L}(X)$ at each vector $x \in X$. Afterwards,

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maps on matrices or on the algebra of all linear and continuous operators preserving local spectrum, local spectral radius, local inner spectral radius have been studied, the authors usually obtaining that they are of a standard form; see for instance the last section of the survey article [3] and the references therein.

For background information on general local spectral theory, we refer to the monographs [1,6,7]. In this paper, we shall consider only the case of maps on the algebra $\mathcal{M}_n$ of all complex $n \times n$ matrices. The form of the local spectrum for linear maps acting on a finite-dimensional space is well-known: a very simple description can be found in [8]. We include it here, for the sake of completeness. (See also [5].)

For $T \in \mathcal{M}_n$, let $\lambda_1, \dots, \lambda_k$, with $1 \le k \le n$, denote the distinct eigenvalues of T. For each eigenvalue λ_i, denote the corresponding root space by N_i; we have $N_i = \ker(T - \lambda_i I_n)^{n_i}$ for some $n_i \in \{1, \dots, n\}$. (By I_n we denote the $n \times n$ identity matrix.) Denote also by $R_i := R((T - \lambda_i I_n)^{n_i})$ the range of the operator $(T - \lambda_i I_n)^{n_i}$ in $\mathbb{C}^n$. Then we can write

$$\mathbb{C}^n = N_1 \oplus \cdots \oplus N_k \tag{1}$$

and $R_i = \oplus_{j \ne i} N_j$ for $i = 1, \dots, k$, both direct sums being algebraic. For $i = 1, \dots, k$, denote by P_i the projection on $\mathbb{C}^n$ with $\ker P_i = R_i$ and $R(P_i) = N_i$. Then $P_i P_j = 0$ for $i \ne j$ and $P_1 + \cdots + P_k = I_n$. For the local spectrum of T, we have that

$$\sigma_T(x) = \{\lambda_i : 1 \le i \le k,\ P_i(x) \ne 0\} \qquad (x \in \mathbb{C}^n). \tag{2}$$

For $T \in \mathcal{M}_n$ and $x \in \mathbb{C}^n$, the local spectral radius of T at x is defined by

$$r_T(x) = \limsup_{j \to \infty} ||T^j(x)||^{1/j}.$$

For $x \ne 0$, as in the case of the classical spectral radius we have that $r_T(x)$ coincides with the maximum modulus of $\sigma_T(x)$. Then using (2),

$$r_T(x) = \max\{|\lambda_i| : 1 \le i \le k,\ P_i(x) \ne 0\} \qquad (x \in \mathbb{C}^n;\ x \ne 0). \tag{3}$$

For $T \in \mathcal{M}_n$ and $x \in \mathbb{C}^n$ nonzero, the local inner spectral radius of T at x is defined by

$$\gamma_T(x) = \min\{|\lambda| : \lambda \in \sigma_T(x)\}.$$

Then (2) gives

$$\gamma_T(x) = \min\{|\lambda_i| : 1 \le i \le k,\ P_i(x) \ne 0\} \qquad (x \in \mathbb{C}^n;\ x \ne 0). \tag{4}$$