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Upper bounds on the Laplacian spread of graphs

Enide Andrade [a,*], Helena Gomes [a,b], María Robbiano [c], Jonnathan Rodríguez [c]

[a] *Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Aveiro, Portugal*
[b] *Departamento de Ciências Exatas e Naturais, Escola Superior de Educação de Viseu, Instituto Politécnico de Viseu, Portugal*
[c] *Departamento de Matemáticas, Universidad Católica del Norte, Antofagasta, Chile*

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The Laplacian spread of a graph G is defined as the difference between the largest and the second smallest eigenvalue of the Laplacian matrix of G. In this work, an upper bound for this graph invariant, that depends on first Zagreb index, is given. Moreover, another upper bound is obtained and expressed as a function of the nonzero coefficients of the Laplacian characteristic polynomial of a graph.

1. Notation and preliminaries

By an (n, m)-graph $G = (\mathcal{V}(G), \mathcal{E}(G))$, for short $G = (\mathcal{V}, \mathcal{E})$, we mean an undirected simple graph on $|\mathcal{V}| = n$ vertices and $m = |\mathcal{E}|$ edges. If $e \in \mathcal{E}$ is the edge connecting

* Corresponding author.
E-mail addresses: enide@ua.pt (E. Andrade), hgomes@ua.pt (H. Gomes), mrobbiano@ucn.cl (M. Robbiano), jrodriguez01@ucn.cl (J. Rodríguez).

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vertices u and v we say that u and v are adjacent and the edge is denoted by $\{u, v\}$. The notation $u \sim v$ means that $\{u, v\} \in \mathcal{E}$. For $u \in \mathcal{V}$ the set of neighbors of u, $N_G(u)$, is the set of vertices adjacent to u. The cardinality of $N_G(u)$, d_u, is called the vertex degree of u. The smallest and largest vertex degrees of G are denoted by δ and Δ, respectively. A graph in which all vertex degrees are equal to p is regular of degree p (or p-regular). The path and the star with n vertices are denoted by P_n and S_n, respectively. A *caterpillar graph* is a tree of order $n \geq 5$ such that by removing all the pendant vertices one obtains a path with at least two vertices. In this context the caterpillar, $T(q_1, \ldots, q_k)$ is obtained from a path P_k, with $k \geq 2$, by associating the central vertex of the star S_{q_i} $(1 \leq i \leq k)$ to the i-th vertex of the path P_k. The adjacency matrix $A(G)$ of a graph G with $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the square matrix of order n, whose (i, i)-entry is equal to 1 if v_i and v_j are adjacent, and 0 otherwise. The Laplacian matrix of G is given by $L(G) = D(G) - A(G)$ where $D(G)$ is the diagonal matrix whose (i, i)-entry is equal to the degree of $v_i \in \mathcal{V}$. This matrix is positive semidefinite and 0 is always a Laplacian eigenvalue whose multiplicity corresponds to the number of connected components of G with $\mathbf{e}$, the all ones vector, as an associated eigenvector. For spectral results on this matrix see, for instance, [6,8]. There are numerous results in the literature concerning upper and lower bounds on the largest eigenvalue of $L(G)$, see [15,19].

If B is a real symmetric matrix, $\beta_i(B)$ (or simply β_i) and σ_B denote the i-th largest eigenvalue of B and the set of eigenvalues of B, respectively. The set of eigenvalues of $L(G)$ is denoted by $\sigma_L(G)$ and called the Laplacian spectrum of G. If β is an eigenvalue of B and $\mathbf{x}$ is one of its eigenvectors the pair $(\beta, \mathbf{x})$ is an eigenpair of B. From now on we represent the Laplacian eigenvalues of G by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. An important result in graph theory, see [19], states that if G has at least one edge, $\Delta + 1 \leq \mu_1$, and if G is connected the equality is attained if and only if $\Delta = n - 1$. Considering G, an upper bound on Laplacian eigenvalues can be easily obtained, $\mu_1 \leq n$, see [2]. Among the most important Laplacian eigenvalues is the algebraic connectivity of G, defined as the second smallest Laplacian eigenvalue μ_{n-1}, [10]. Recently, the algebraic connectivity has received much attention, see [1,19,22,23] and the references cited therein. A graph is connected if and only if $\mu_{n-1} > 0$, [10].

2. The Laplacian spread of an arbitrary graph and some motivation

Let B be an $n \times n$ complex matrix with eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$. The spread of B (or matricial spread) is defined by

$$s(B) = \max_{i,j} |\beta_i - \beta_j|,$$

where the maximum is taken over all pairs of eigenvalues of B. There is a considerable literature on this parameter, see for instance [13,18,21]. Suppose that $B \in \mathbb{C}^{n \times n}$ is a Hermitian complex matrix. For $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, we denote by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^* \mathbf{y}$, the inner product in $\mathbb{C}^n$ and by $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ the norm of $\mathbf{x}$. Here, $|B| = \sqrt{trace(B^* B)}$ is the Frobenius