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Diagonal Riccati stability and applications

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ABSTRACT

We consider the question of diagonal Riccati stability for a pair of real matrices A, B. A necessary and sufficient condition for diagonal Riccati stability is derived and applications of this to two distinct cases are presented. We also describe some motivations for this question arising in the theory of generalised Lotka–Volterra systems.

1. Introduction and preliminaries

We consider the following problem. Given $A, B \in \mathbb{R}^{n\times n}$, determine conditions for the existence of *diagonal* positive definite matrices P, Q in $\mathbb{R}^{n\times n}$ such that

$$A^T P + PA + Q + PBQ^{-1}B^T P \prec 0, \tag{1}$$

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where $M \prec 0$ denotes that $M = M^T$ is negative definite. Throughout the paper, $M \succ 0$ ($M \succeq 0$) denotes that M is positive definite (positive semi-definite); $M \preceq 0$ denotes that M is negative semi-definite.

When diagonal positive definite solutions P, Q of (1) exist, we say that the pair A, B is *diagonally Riccati stable.*

Our interest in the question stems from the stability of time-delay systems. Specifically, when such a pair P, Q exists, the linear time-delay system associated with A, B admits a Lyapunov–Krasovskii functional of a particularly simple form [8]. The more general question of when positive definite solutions of (1), not necessarily diagonal, exist was highlighted in [17] and some preliminary results on this question were also described in this reference.

We now introduce some notation and terminology, and recall some basic results that will be needed later in the paper.

A matrix A in $\mathbb{R}^{n\times n}$ is *Metzler* if its off-diagonal elements are nonnegative: formally, $a_{ij} \geq 0$ for $i \neq j$. As is standard, a *nonnegative* matrix A is one satisfying $a_{ij} \geq 0$ for $1 \leq i, j \leq n$. For vectors v, w in $\mathbb{R}^n$, $v \geq w$ is understood componentwise and means that $v_i \geq w_i$ for $1 \leq i \leq n$. Similarly, $v > w$ means $v \geq w$, $v \neq w$ and $v \gg w$ means $v_i > w_i$ for $1 \leq i \leq n$. Similar notation is used for matrices of compatible dimensions. In particular, $A \geq 0$, ($A \leq 0$) denotes that A is entrywise nonnegative (nonpositive).

For a positive integer n, we denote by $Sym(n, \mathbb{R})$ the space of $n \times n$ real symmetric matrices. For $A \in \mathbb{R}^{n\times n}$, A^T denotes the transpose of A. A matrix $A \in \mathbb{R}^{n\times n}$ is *Hurwitz* if all of its eigenvalues have negative real parts. It is classical that A is Hurwitz if and only if there exists some $P \succ 0$ with $A^T P + PA \prec 0$.

The following result recalls some well-known facts concerning Metzler matrices [11].

Proposition 1.1. *Let $A \in \mathbb{R}^{n\times n}$ be Metzler. The following are equivalent:*

(i) *A is Hurwitz;*
(ii) *there exists some vector $v \gg 0$ in $\mathbb{R}^n$ with $Av \ll 0$;*
(iii) *there exists some positive definite diagonal matrix $D \in \mathbb{R}^{n\times n}$ with $A^T D + DA \prec 0$;*
(iv) *for every non-zero $v \in \mathbb{R}^n$, there is some index i with $v_i(Av)_i < 0$;*
(v) *$A^{-1} \leq 0$.*

The next result concerning the Riccati equation is based on the Schur complement and follows from Theorem 7.7.6 in [10].

Lemma 1.1. *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ be given. Then the matrices $P \succ 0$, $Q \succ 0$ in $\mathbb{R}^{n\times n}$ satisfy (1) if and only if:*

$$S := \begin{pmatrix} A^T P + PA + Q & PB \\ B^T P & -Q \end{pmatrix} \prec 0. \tag{2}$$