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The spectral radius of edge chromatic critical graphs

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A B S T R A C T

A connected graph G with maximum degree Δ and edge chromatic number $\chi'(G) = \Delta + 1$ is called Δ-critical if $\chi'(G-e) = \Delta$ for every edge e of G. In this paper, we consider two weaker versions of Vizing's conjecture, which concern the spectral radius $\rho(G)$ and the signless Laplacian spectral radius $\mu(G)$ of G. We obtain some lower bounds for $\rho(G)$ and $\mu(G)$, and present some cases where the conjectures are true. Finally, several open problems are also proposed.

1. Introduction

We consider simple connected graphs in this paper. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$, edge set $E(G)$, with $|V(G)| = n$, $|E(G)| = m$. For a vertex x,

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we set $N(x) = \{v : xv \in E(G)\}$ and $d(x) = d_G(x) = |N(x)|$, the degree of x in G. The maximum and minimum degrees of G are denoted by $\Delta(G) = \Delta$ and $\delta(G) = \delta$, respectively. A vertex of maximum degree in G is called a *major vertex*. We use $d_\Delta(x)$ to denote the number of major vertices of G adjacent to x.

The *adjacency matrix* of a graph G is $A(G) = (a_{ij})$, where $a_{ij} = 1$ if two vertices i and j are adjacent in G, and $a_{ij} = 0$ otherwise. Let $D(G) = (d_{ij})$ be the diagonal degree matrix of G, i.e., d_{ii} is the degree of the vertex i in G, and $d_{ij} = 0$ otherwise. We call the matrix $L(G) = D(G) - A(G)$ the *Laplacian matrix* of G, and the matrix $Q(G) = D(G) + A(G)$ the *signless Laplacian matrix* or *Q-matrix* of G. We denote the largest eigenvalues of $A(G)$ and $Q(G)$ by $\rho(G)$ and $\mu(G)$, respectively, and call them *the spectral radius* and *the signless Laplacian spectral radius* (or the *Q-spectral radius*) of G, respectively. For background on the matrices $A(G)$ and $Q(G)$ of G, the reader is referred to [6,7] and the references therein.

Brualdi and Solheid [1] proposed the following problem concerning the spectral radius of graphs: *Given a set $\mathcal{G}$ of graphs, find an upper bound for the spectral radius over all graphs of $\mathcal{G}$, and characterize the graphs in which the supremum spectral radius is attained.* Inspired by this problem, the eigenvalues of special classes of graphs are well studied in the literature, such as graphs with given chromatic number [9], matching number [10], diameter [11], and domination number [28]. One can refer to a recent, comprehensive book by Stevanović [29] for more details. Also, the theory of eigenvalues of graphs has found successful applications in other disciplines such as chemistry and biology, and a typical and widely studied invariant is the energy of graphs, for which one may refer to [16–19] and the references therein.

A *k-edge-coloring* of a graph G is a function $\phi : E(G) \to \{1, \ldots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges e and e'. That is, if we consider $\{1, \ldots, k\}$ as a set of k colors, then any two adjacent edges receive different colors. The *edge chromatic number* of G, denoted by $\chi'(G)$, is the smallest integer k such that G has a k-edge-coloring. The celebrated Vizing's Theorem [31] states that $\chi'(G)$ is either Δ or $\Delta + 1$. A graph G is *class one* if $\chi'(G) = \Delta$ and *class two* if $\chi'(G) = \Delta + 1$. A class two graph G is Δ-*critical* if $\chi'(G - e) = \Delta$ for each edge e of G and it has maximum degree Δ. We simply say that G is *critical* if we do not wish to refer to its maximum degree Δ.

The theory of edge-colorings in graphs is one of the most fundamental areas in graph theory, and often appears in various scheduling problems like the file transfer problem on computer networks. The main tools in previous research of edge-coloring problems include Vizing's Adjacency Lemma, the discharging method, the Vizing Fans, the Kierstead Paths and the Tashkinov Trees. One may refer to the monograph [30] for details. Thus according to the Brualdi–Solheid problem, it would be interesting to consider the spectral properties of the class two graphs, specifically, the edge chromatic critical graphs.

Vizing [31] proposed the following conjecture on the size of critical graphs.