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Power sum decompositions of elementary symmetric polynomials

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We bound the tensor ranks of elementary symmetric polynomials, and we give explicit decompositions into powers of linear forms. The bound is attained when the degree is odd.

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1. Introduction

Given a form $F \in \mathbb{C}[x_1, \ldots, x_n]$ of degree d, the *symmetric tensor rank* (rank in short) of F is the least integer s such that $f = \sum_{i=1}^{s} L_i^d$, where the L_i's are linear forms. For a generic form, the rank is known for any n and d by a work of Alexander and Hirschowitz [1], and a simple proof is proposed by Chandler [4].

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On the contrary, only a few cases are known for that of specific forms [2,3,8,9]. To describe one of them, we define an index-membership function δ; for an integer set I and an integer i, define $\delta(I, i) = -1$ if $i \in I$, or 1 otherwise. A decomposition of the monic square-free monomial $\sigma_{n,n}$ in n variables was given by Fischer [6]. He showed that

$$2^{n-1} n! \cdot \sigma_{n,n} = \sum_{I \subset [n] \setminus \{1\}} (-1)^{|I|} (x_1 + \delta(I, 2)x_2 + \cdots + \delta(I, n)x_n)^n, \tag{1.1}$$

where $[n]$ denotes the set $\{1, 2, \ldots, n\}$, which will be used though this paper. For a general monomial, such a decomposition is given in [2, Corollary 3.8] so that (1.1) is a special case.

We extend the decomposition (1.1) for general elementary symmetric polynomials $\sigma_{d,n}$. As an example, we have

$$\begin{aligned} 24\sigma_{3,5}(a, b, c, d, e) &= abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde \\ &= 3(a + b + c + d + e)^3 - (-a + b + c + d + e)^3 \\ &\quad - (a - b + c + d + e)^3 - (a + b - c + d + e)^3 \\ &\quad - (a + b + c - d + e)^3 - (a + b + c + d - e)^3, \end{aligned}$$

and it gives an upper bound $\operatorname{rank}(\sigma_{3,5}) \leq 6$.

In general, such a decomposition provides an upper bound

$$\operatorname{rank}(\sigma_{d,n}) \leq \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{n}{i}$$

for the rank of $\sigma_{d,n}$ (Corollary 2.2, Corollary 4.4).

This paper is organized as follows. In section 2 we present a power sum decomposition of $\sigma_{d,n}$ for odd degree case. Theorem 2.1 gives an analogue of (1.1).

In Section 3 we observe a structure of the catalecticants of $\sigma_{d,n}$. Lemma 3.2 says that each catalecticant matrix is essentially full rank, in the sense that it can be refined to a full rank matrix after removing all zero rows and zero columns. Theorem 3.4 tells us that the lower bound derived by Lemma 3.2 matches the number of components in the decomposition given in Section 2, hence we get the rank of $\sigma_{d,n}$ for odd d.

Section 4 discusses the even degree case. A power sum decomposition of $\sigma_{d,n}$ can be obtained from that of $\sigma_{d+1,n}$ (Theorem 4.1), or it can be derived directly (Remark 4.2). By Corollary 4.4 we see that upper and lower bounds are not the same, and we shortly explain why it seems to be hard to improve these bounds.

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