Linear Algebra and its Applications 492 (2016) 98–110

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A generalization of the Avram–Parter Theorem

A. Bourget *, C. Espinosa, S. Faulkinbury

Department of Mathematics, California State University, Fullerton, United States

ARTICLE INFO

Article history:
Received 15 October 2015
Accepted 13 November 2015
Available online 23 November 2015
Submitted by V. Mehrmann

MSC:
47B36
34L20

Keywords:
Singular values
Avram–Parter Theorem
Asymptotic distribution

ABSTRACT

The First Szegő Limit Theorem gives a remarkable connection between the eigenvalues distribution of a Hermitian Toeplitz matrix and its symbol. This result was extended by Avram and Parter to the singular values of complex Toeplitz matrices. The purpose of this note is to extend their results to a larger class of matrices whose entries are equidistributed and have small mean variation. We also present applications to Kac–Murdock–Szegő matrices, block Toeplitz and locally Toeplitz matrices.

1. Introduction

Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$ and let $a \in L^2(\mathbb{T})$. The Toeplitz matrix $T_n(a)$ of order n with symbol a is defined as the matrix

$$T_n(a) = [\hat{a}_{j-i}]_{i,j=1}^n$$

* Corresponding author. Mailing address: Department of Mathematics, California State University (Fullerton), McCarthy Hall 154, Fullerton, CA 92834, United States.

E-mail address: abourget@fullerton.edu (A. Bourget).

http://dx.doi.org/10.1016/j.laa.2015.11.016

where $\hat{a}_k$ is the kth Fourier coefficient of a, i.e.

$$\hat{a}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(t) e^{-ikt} \, dt.$$

The Avram–Parter Theorem [4,2] is concerned with the asymptotic distribution of the singular values $\sigma_{k;n}$ of $T_n(a)$. It states

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(\sigma_{k,n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(|a(t)|) \, dt$$

for every continuous function φ on $\mathbb{R}^+ = [0, \infty)$. This result was first obtained by Parter [11] for bounded continuous symbols and then extended by Avram [1] to symbols in $L^\infty(\mathbb{T})$. The above version with symbols in $L^2(\mathbb{T})$ is due to Tyrtyshnikov and Zamarashkin [16]. Other extensions were obtained for the block-matrix case [13,15] and the locally Toeplitz case [14,12].

In this paper, we extend the Avram–Parter Theorem to a class of non-Toeplitz matrices. Namely, we consider sequences of matrices whose diagonals satisfy a vanishing mean variation condition and whose entries are equidistributed with respect to some probability measure on compact subsets of $l^2(\mathbb{C})$.

In the next section, we recall some definitions and state our main result – the proof is given in the third section of the paper. We conclude with some applications to Kac–Murdock–Szegő matrices, block matrices and locally Toeplitz matrices.

2. Main result

For any non-negative integers m and n, let $m \wedge n = \min\{m, n\}$ and let $\mathcal{M}_{m,n}(\mathbb{C})$ be the set of all m by n matrices with complex entries. We denote the singular values of $A_{m,n} \in \mathcal{M}_{m,n}(\mathbb{C})$ by

$$\sigma_1(A_{m,n}) \geq \sigma_2(A_{m,n}) \geq \cdots \geq \sigma_{m \wedge n}(A_{m,n}) \geq 0.$$

Recall, the Frobenius and operator norms of $A_{m,n}$ are respectively given by

$$\|A_{m,n}\|_F = \left(\sum_{k=1}^{m \wedge n} \sigma_k^2(A_{m,n}) \right)^{1/2} \qquad \text{and} \qquad \|A_{m,n}\|_\infty = \sigma_1(A_{m,n}).$$

In order to compute the limiting statistical distribution (LSD) of the singular values of sequences of matrices $\{A_{m,n}\}$, it is customary to impose some restrictions on the growth of the entries a_{ij} of $A_{m,n}$. Here, we make the assumption