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On the distance and distance Laplacian eigenvalues of graphs

Huiqiu Lin [a,*], Baoyindureng Wu [b], Yingying Chen [c], Jinlong Shu [c]

[a] *Department of Mathematics, East China University of Science and Technology, Shanghai, PR China*
[b] *College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, PR China*
[c] *Department of Mathematics, East China Normal University, Shanghai, PR China*

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A B S T R A C T

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Let $D(G)$ be the distance matrix of G. For a given nonnegative integer k, when n is sufficiently large with respect to k, we show that $\lambda_{n-k}(D) \le -1$, thereby solving a problem proposed by Lin et al. (2014) [8]. The distance Laplacian spectral radius of a connected graph G is the spectral radius of the distance Laplacian matrix of G, defined as

$$D^L(G) = Tr(G) - D(G),$$

where $Tr(G)$ is the diagonal matrix of vertex transmissions of G. Aouchiche and Hansen (2014) [3] conjectured that $m(\lambda_1(D^L)) \le n - 2$ when $G \ncong K_n$, and the equality holds if and only if either $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$. In this paper, we confirm the conjecture.

* Corresponding author.
E-mail address: huiqiulin@126.com (H. Lin).

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1. Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$, $|E(G)| = m$. Also let d_i be the degree of the vertex $v_i \in V(G)$. We always assume that the graph under consideration is connected when the problem is concerned with the distance. Let $A = (a_{ij})$ be the adjacency matrix of G where $a_{ij} = 1$ if $v_i v_j \in E(G)$ and $a_{ij} = 0$, otherwise. We use $\lambda_i(G)$ to denote the eigenvalues of A. Let $G = (V, E)$ be a regular graph with n vertices and degree k. As usual, we use K_n and P_n to denote the complete graph and the path on n vertices, respectively. G is said to be strongly regular with parameters (n, k, a, c) if there are also integers a and c such that every two adjacent vertices have a common neighbors and every two non-adjacent vertices have c common neighbors. The distance between vertex v_i and v_j, denoted by $d_G(v_i, v_j)$, is the length of a shortest path from v_i to v_j. The diameter of a graph is the maximum distance between any two vertices of G. Let d be the diameter of G. The transmission $Tr(v_i)$ (or D_i) of vertex v_i is defined to be the sum of distances from v_i to all other vertices, that is,

$$Tr(v_i) = \sum_{v_j \in V(G)} d(v_i, v_j).$$

The distance matrix of G, denoted by $D(G)$ or simply by D, is the symmetric real matrix with (i, j)-entry being $d_G(v_i, v_j)$ (or d_{ij}). Let $\lambda_1(D(G)) \geq \lambda_2(D(G)) \geq \cdots \geq \lambda_n(D(G))$ denote the spectra of D. The research for distance matrix can be dated back to the papers [4,5], which presented an interesting result that the determinant of the distance matrix of trees with order n is always $(-1)^{n-1}(n-1)2^{n-2}$, independent of the structure of the tree. Recently, the distance matrix of a graph has received increasing attention, see [6,7,12]. For more results on the distance matrix, you can refer to the excellent survey [1].

Lin et al. [8] showed that for a connected graph G of order n, $\lambda_{n-1}(D) \leq -1$ if $n \geq 4$ and $\lambda_{n-2}(D) \leq -1$ if $n \geq 7$. Furthermore, they proposed the following problem.

Problem 1. Given an integer k, when n is sufficiently large with respect to k, for any connected graph G of order n, does $\lambda_{n-k}(D) \leq -1$?

By establishing the following theorem, we answer the above problem in positive.

Theorem 1.1. *Let D be the distance matrix of a connected graph G of order n. For a non-negative integer k, $\lambda_{n-k}(D) \leq -1$ provided n is sufficiently large with respect to k.*

Obviously, $Tr(v_i)$ is the sum of i-th row of $D(G)$. Let $Tr(G) = \mathrm{diag}(Tr_G(v_1), Tr_G(v_2), \ldots, Tr_G(v_n))$ be the diagonal matrix of vertex transmissions of G. Similar to the Laplacian matrix of a graph, the distance Laplacian matrix of G is introduced by Aouchiche and Hansen [2],