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# Estimability of variance components when all model matrices commute

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This paper deals with estimability of variance components in mixed models when all model matrices commute. In this situation, it is well known that the best linear unbiased estimators of fixed effects are the ordinary least squares estimators. If, in addition, the family of possible variance–covariance matrices forms an orthogonal block structure, then there are the same number of variance components as strata, and the variance components are all estimable if and only if there are non-zero residual degrees of freedom in each stratum. We investigate the case where the family of possible variance–covariance matrices, while still commutative, no longer forms an orthogonal block structure. Now the variance components may or may not all be estimable, but there is no clear link with residual degrees of freedom. Whether or not they are all estimable, there may or may not be uniformly best unbiased

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quadratic estimators of those that are estimable. Examples are given to demonstrate all four possibilities.


## 1. Some assumptions about the linear model

Let $\mathbf{Y}$ be a column vector of $N$ random variables $Y_1, \ldots, Y_N$. Write $\mathbb{E}(\mathbf{Y})$ for the expectation of the vector $\mathbf{Y}$, and $\mathbf{V}$ for its variance–covariance matrix. In this section we present some of the assumptions that are commonly made about $\mathbb{E}(\mathbf{Y})$ and $\mathbf{V}$ in order to have a linear model with good properties.

**Assumption 1** *(Linear expectation).* There is a known integer $n$, a known $N \times n$ real matrix $\mathbf{X}$ and an unknown column vector $\boldsymbol{\tau}$ of length $n$ such that $\mathbb{E}(\mathbf{Y}) = \mathbf{X}\boldsymbol{\tau}$.

Under Assumption 1, let $\mathbf{T}$ be the $N \times N$ matrix of orthogonal projection onto the column-space of $\mathbf{X}$. Then $\mathbf{T} = \mathbf{X}(\mathbf{X}^\top\mathbf{X})^+\mathbf{X}^\top$, where $^+$ denotes the Moore–Penrose generalized inverse: see texts such as [11,23]. Also, let $\mathbf{G}_N$ be the matrix of orthogonal projection onto the space $W$ spanned by the all-1 vector $\mathbf{1}$, so that $\mathbf{G}_N = N^{-1}\mathbf{J}_N$, where $\mathbf{J}_N$ is the $N \times N$ matrix whose entries are all equal to 1. It is often the case that $\mathbf{1}$ is in the column-space of $\mathbf{X}$. This happens if and only if $\mathbf{T}\mathbf{G}_N = \mathbf{G}_N\mathbf{T} = \mathbf{G}_N$: see [11,39].

**Assumption 2** *(Spectral form of variance–covariance matrix).* There are known orthogonal symmetric idempotent matrices $\mathbf{Q}_0, \ldots, \mathbf{Q}_m$ summing to the identity matrix $\mathbf{I}_N$ of order $N$, and non-negative scalars $\gamma_0, \ldots, \gamma_m$ such that

$$\mathbf{V} = \sum_{i=0}^{m} \gamma_i \mathbf{Q}_i. \tag{1}$$

This assumption says that the scalars $\gamma_0, \ldots, \gamma_m$ are the eigenvalues of $\mathbf{V}$; the corresponding eigenspaces are the column spaces of $\mathbf{Q}_0, \ldots, \mathbf{Q}_m$, and these are known. It is often the case that $W$ is one of the eigenspaces: in that case, we label the spaces so that $\mathbf{Q}_m = \mathbf{G}_N$.

**Assumption 3** *(No relations among the eigenvalues).* Assumption 2 is true and there are no further constraints on the values of $\gamma_0, \ldots, \gamma_m$.

A weaker form of this, given in [37], is that there are no linear constraints on $\gamma_0, \ldots, \gamma_m$. Here is an alternative weaker form.

**Assumption 4** *(Cone of full dimension).* Assumption 2 is true and the family of possible matrices $\mathbf{V}$ forms a positive cone of dimension $m+1$ in the space spanned by $\mathbf{Q}_0, \ldots, \mathbf{Q}_m$.