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A multivariate generalization of Prony's method

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A B S T R A C T

Prony's method is a prototypical eigenvalue analysis based method for the reconstruction of a finitely supported complex measure on the unit circle from its moments up to a certain degree. In this note, we give a generalization of this method to the multivariate case and prove simple conditions under which the problem admits a unique solution. Provided the order of the moments is bounded from below by the number of points on which the measure is supported as well as by a small constant divided by the separation distance of these points, stable reconstruction is guaranteed. In its simplest form, the reconstruction method consists of setting up a certain multilevel Toeplitz matrix of the moments, compute a basis of its kernel, and compute by some method of choice the set of common roots of the multivariate polynomials whose coefficients are given in the second step. All theoretical results are illustrated by numerical experiments.

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1. Introduction

In this paper we propose a generalization of de Prony's classical method [10] for the parameter and coefficient reconstruction of univariate finitely supported complex measures to a finite number of variables. The method of de Prony lies at the core of seemingly different classes of problems in signal processing such as spectral estimation, search for an annihilating filter, deconvolution, spectral extrapolation, and moment problems. Thus we provide a new tool to analyze multivariate versions of a broad set of problems.

To recall the machinery of the classical Prony method let $\mathbb{C}_* := \mathbb{C}\setminus\{0\}$ and let $\hat{f}_j \in \mathbb{C}_*$ and pairwise distinct $z_j \in \mathbb{C}_*$, $j = 1, \dots, M$, be given. Let δ_{z_j} denote the Dirac measure in z_j on $\mathbb{C}_*$ and let

$$\mu = \sum_{j=1}^{M} \hat{f}_j \delta_{z_j}$$

be a finitely supported complex measure on $\mathbb{C}_*$. By the Prony method the $\hat{f}_j$ and z_j are reconstructed from $2M + 1$ moments

$$f(k) = \int_{\mathbb{C}_*} x^k \mathrm{d}\mu(x) = \sum_{j=1}^{M} \hat{f}_j z_j^k, \quad k = -M, \dots, M.$$

Since the coefficients $\hat{p}_\ell \in \mathbb{C}$, $\ell = 0, \dots, M$, of the (not a priori known) so-called Prony polynomial

$$p(Z) := \prod_{j=1}^{M} (Z - z_j) = \sum_{\ell=0}^{M} \hat{p}_\ell Z^\ell$$

fulfill the linear equations

$$\sum_{\ell=0}^{M} \hat{p}_\ell f(\ell - m) = \sum_{j=1}^{M} \hat{f}_j z_j^{-m} \sum_{\ell=0}^{M} \hat{p}_\ell z_j^\ell = \sum_{j=1}^{M} \hat{f}_j z_j^{-m} p(z_j) = 0, \quad m = 0, \dots, M,$$

and are in fact the unique solution to this system with $\hat{p}_M = 1$, reconstruction of the z_j is possible by computing the kernel vector $(\hat{p}_1, \dots, \hat{p}_{M-1}, 1)$ of the rank-M Toeplitz matrix

$$T := (f(k - \ell))_{\substack{\ell=0,\dots,M \\ k=0,\dots,M}} \in \mathbb{C}^{M+1 \times M+1}$$

and, knowing this to be the coefficient vector of p, compute the roots z_j of p. Afterwards, the coefficients $\hat{f}_j$ of f (that did not enter the discussion until now) can be uniquely recovered by solving a Vandermonde linear system of equations. When attempting to