Linear Algebra and its Applications 490 (2016) 102–123

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Simple graded division algebras over the field of real numbers ☆

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ARTICLE INFO

Article history:
Received 27 June 2015
Accepted 19 October 2015
Available online 11 November 2015
Submitted by R. Brualdi

MSC:
primary 17B70
secondary 16R60, 16W50, 17B60

Keywords:
Graded algebra
Simple Lie algebra
Grading
Primitive algebra
Functional identity

ABSTRACT

We classify, up to equivalence, all finite-dimensional simple graded division algebras over the field of real numbers. The grading group is any finite abelian group.

1. Introduction

A unital algebra R over a field F graded by a group G is called *graded division* if every nonzero homogeneous element is invertible. Clearly, each such algebra is graded

☆ The first author acknowledges support by NSERC grant # 227060-09. The second author acknowledges support by RFBR grant # 13-01-00234a.

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http://dx.doi.org/10.1016/j.laa.2015.10.021
0024-3795/

simple, that is, R has non-nonzero graded ideals. In the classification of gradings on simple finite-dimensional algebras one is interested in graded division algebras which are simple at the same time. Indeed, according to graded analogues of Schur's Lemma and Density Theorem (see, for example, [8, Chapter 1]) any such algebra is isomorphic to the algebra $\mathrm{End}_D V$ of endomorphisms of a finite-dimensional graded (right) vector space over a graded division algebra D. Since R is simple, it is obvious that D must be simple, as well.

In the case where the field F is algebraically closed and the group G is finite abelian, all such graded division algebras have been described in [1] and [5] and used in [4] and [3] to classify gradings on simple finite-dimensional Lie algebras over algebraically closed fields. For full account see [8, Chapter 1], where the authors treat also the case of Artinian algebras. In [2] (see also [7], for a particular case) the authors treat the case of primitive algebras with minimal one-sided ideals. If such algebras are locally finite, the graded division algebras arising are finite-dimensional and so the description provided in the case of finite-dimensional algebras works in this situation, as well.

The main results of this paper are Theorem 3.1 and Theorem 7.3. In Theorem 3.1 we list all equivalence classes of division gradings on simple finite-dimensional real associative algebras. In Theorem 7.3 we apply these results to the classification of all abelian group gradings on such algebras. A special feature of the real case is the usage of Clifford algebra, which provides a natural approach to the study of gradings in the case of real algebras.

2. Preliminaries

A vector decomposition $\Gamma : V = \bigoplus_{g\in G} V_g$ is called a grading of a vector space V over a field F by a set G. The subset S of all $s \in G$ such that $V_s \neq \{0\}$ is called the *support* of Γ and is denoted by $\mathrm{Supp}\,\Gamma$ (also as $\mathrm{Supp}\,V$, if S is endowed just by one grading). If $\Gamma' : V' = \bigoplus_{g'\in G'} V'_{g'}$ is a grading of another space then a homomorphism of gradings $\varphi : \Gamma \to \Gamma'$ is a linear map $f : V \to V'$ such that for each $g \in G$ there exists (unique) $g' \in G'$ such that $\varphi(V_g) \subset V_{g'}$. If φ has an inverse as homomorphism of grading then we say that $\varphi : \Gamma \to \Gamma'$ is an *equivalence* of gradings Γ and Γ' (or graded vector spaces V and V').

A grading $\Gamma : R = \bigoplus_{g\in G} R_g$ of an algebra R over a field F is an *algebra* grading if for any $s_1, s_2 \in \mathrm{Supp}\,\Gamma$ such that $R_{s_1}R_{s_2} \neq \{0\}$ there is $s_3 \in G$ such that $R_{s_1}R_{s_2} \subset R_{s_3}$. Two algebra gradings $\Gamma : R = \bigoplus_{g\in G} R_g$ and $\Gamma' : R' = \bigoplus_{g\in G'} R'_{g'}$ of algebras over a field $\mathbb{F}$ are called *equivalent* if there exists an algebra isomorphism $\varphi : R \to R'$, which is an equivalence of vector space gradings. In this case there is a bijection $\alpha : \mathrm{Supp}\,\Gamma \to \mathrm{Supp}\,\Gamma'$ such that $\varphi(R_g) = R'_{\alpha(g)}$.

If G is a group then a grading $\Gamma : R = \bigoplus_{g\in G} R_g$ of an algebra R over a field F is called a *group grading* if for any $g, h \in G$, we have $R_g R_h \subset R_{gh}$. Normally, it is assumed that the grading group G is generated by $\mathrm{Supp}\,\Gamma$. If $\varphi : \Gamma \to \Gamma' : R \to R'$ is an equivalence of gradings of algebras R and R' by groups G and G' and the accompanying bijection