

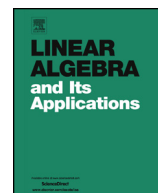


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Some operator and trace function convexity theorems



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ABSTRACT

We consider trace functions $(A, B) \mapsto \text{Tr}[(A^{q/2} B^p A^{q/2})^s]$ where A and B are positive $n \times n$ matrices and ask when these functions are convex or concave. We also consider operator convexity/concavity of $A^{q/2} B^p A^{q/2}$ and convexity/concavity of the closely related trace functional $\text{Tr}[A^{q/2} B^p A^{q/2} C^r]$. The concavity questions are completely resolved, thereby settling cases left open by Hiai; the convexity questions are settled in many cases. As a consequence, the Audenaert–Datta Rényi entropy conjectures are proved for some cases.

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1. Introduction

Let \mathcal{P}_n denote the set of $n \times n$ positive definite matrices. For $p, q, s \in \mathbb{R}$, define

$$\Phi_{p,q,s}(A, B) = \text{Tr}[(A^{q/2} B^p A^{q/2})^s]. \quad (1.1)$$

We are mainly interested in the convexity or concavity of the map $(A, B) \mapsto \Phi_{p,q,s}(A, B)$, but we are also interested in the *operator* convexity/concavity of $A^{q/2} B^p A^{q/2}$. When any of p, q or s is zero, the question of convexity is trivial, and we exclude these cases.

Given any $n \times n$ matrix K , and with p, q, s as above, define

$$\Psi_{K,p,q,s}(A, B) = \text{Tr}[(A^{q/2} K^* B^p K A^{q/2})^s], \quad (1.2)$$

and note that

$$\Phi_{p,q,s}(A, B) = \Psi_{\mathbb{1},p,q,s}(A, B). \quad (1.3)$$

The main question to be addressed here is this: *For which non-zero values of p, q and s is $\Psi_{K,p,q,s}(A, B)$ jointly convex or jointly concave on $\mathcal{P}_n \times \mathcal{P}_n$ for all n and all K ?*

We begin with several simple reductions. Since invertible K are dense, it suffices to consider all invertible operators K . Then, for K invertible,

$$\Psi_{K,p,q,s}(A, B) = \Psi_{(K^*)^{-1}, -p, -q, -s}(A, B),$$

and therefore it is no loss of generality to assume that $s > 0$. We always make this assumption in what follows.

Next, the convexity/concavity properties of $\Psi_{K,p,q,s}(A, B)$ are a consequence of those of $\Phi_{p,q,s}(A, B)$, and hence it suffices to study the special case $K = \mathbb{1}$. In fact, more is true as stated in the following [Lemma 1.1](#). These equivalences may be useful in other contexts. (For $s = 1$ the equivalence of (1) and (4) is in [\[11\]](#) and the equivalence of (1) and (3) is in [\[4\]](#); the arguments in those papers extend to all s , but we repeat them here for completeness.)

Lemma 1.1 (*Equivalent formulations*). *The following statements are equivalent for fixed p, q, s .*

- (1) *The map $(A, B) \mapsto \Psi_{K,p,q,s}(A, B)$ is convex for all K and all n .*
- (2) *The map $(A, B) \mapsto \Psi_{K,p,q,s}(A, B)$ is convex for all unitary K and all n .*
- (3) *The map $(A, B) \mapsto \Psi_{\mathbb{1},p,q,s}(A, B) = \Phi_{p,q,s}(A, B)$ is convex for all n .*
- (4) *The map $A \mapsto \Psi_{K,p,q,s}(A, A)$ is convex for all K and all n .*
- (5) *The map $A \mapsto \Psi_{K,p,q,s}(A, A)$ is convex for all unitary K and all n .*

The same is true if convex is replaced by concave in all statements.

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