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Some operator and trace function convexity theorems

Eric A. Carlen [a,*,1], Rupert L. Frank [b,2], Elliott H. Lieb [c,3]

[a] *Department of Mathematics, Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, United States*
[b] *Department of Mathematics, Caltech, Pasadena, CA 91125, United States*
[c] *Departments of Mathematics and Physics, Jadwin Hall, Princeton University, Washington Road, Princeton, NJ 08544, United States*

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We consider trace functions $(A, B) \mapsto \mathrm{Tr}[(A^{q/2}B^pA^{q/2})^s]$ where A and B are positive $n \times n$ matrices and ask when these functions are convex or concave. We also consider operator convexity/concavity of $A^{q/2}B^pA^{q/2}$ and convexity/concavity of the closely related trace functional $\mathrm{Tr}[A^{q/2}B^pA^{q/2}C^r]$. The concavity questions are completely resolved, thereby settling cases left open by Hiai; the convexity questions are settled in many cases. As a consequence, the Audenaert–Datta Rényi entropy conjectures are proved for some cases.

* Corresponding author.
E-mail address: carlen@math.rutgers.edu (E.A. Carlen).

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1. Introduction

Let $\mathcal{P}_n$ denote the set of $n \times n$ positive definite matrices. For $p, q, s \in \mathbb{R}$, define

$$\Phi_{p,q,s}(A, B) = \mathrm{Tr}[(A^{q/2} B^p A^{q/2})^s] \; . \tag{1.1}$$

We are mainly interested in the convexity or concavity of the map $(A, B) \mapsto \Phi_{p,q,s}(A, B)$, but we are also interested in the *operator* convexity/concavity of $A^{q/2} B^p A^{q/2}$. When any of p, q or s is zero, the question of convexity is trivial, and we exclude these cases.

Given any $n \times n$ matrix K, and with p, q, s as above, define

$$\Psi_{K,p,q,s}(A, B) = \mathrm{Tr}[(A^{q/2} K^* B^p K A^{q/2})^s] \; , \tag{1.2}$$

and note that

$$\Phi_{p,q,s}(A, B) = \Psi_{\mathbb{1},p,q,s}(A, B) \; . \tag{1.3}$$

The main question to be addressed here is this: *For which non-zero values of p, q and s is $\Psi_{K,p,q,s}(A, B)$ jointly convex or jointly concave on $\mathcal{P}_n \times \mathcal{P}_n$ for all n and all K?*

We begin with several simple reductions. Since invertible K are dense, it suffices to consider all invertible operators K. Then, for K invertible,

$$\Psi_{K,p,q,s}(A, B) = \Psi_{(K^*)^{-1},-p,-q,-s}(A, B) \; ,$$

and therefore it is no loss of generality to assume that $s > 0$. We always make this assumption in what follows.

Next, the convexity/concavity properties of $\Psi_{K,p,q,s}(A, B)$ are a consequence of those of $\Phi_{p,q,s}(A, B)$, and hence it suffices to study the special case $K = \mathbb{1}$. In fact, more is true as stated in the following Lemma 1.1. These equivalences may be useful in other contexts. (For $s = 1$ the equivalence of (1) and (4) is in [11] and the equivalence of (1) and (3) is in [4]; the arguments in those papers extend to all s, but we repeat them here for completeness.)

Lemma 1.1 *(Equivalent formulations). The following statements are equivalent for fixed p, q, s.*

(1) The map $(A, B) \mapsto \Psi_{K,p,q,s}(A, B)$ is convex for all K and all n.
(2) The map $(A, B) \mapsto \Psi_{K,p,q,s}(A, B)$ is convex for all unitary K and all n.
(3) The map $(A, B) \mapsto \Psi_{\mathbb{1},p,q,s}(A, B) = \Phi_{p,q,s}(A, B)$ is convex for all n.
(4) The map $A \mapsto \Psi_{K,p,q,s}(A, A)$ is convex for all K and all n.
(5) The map $A \mapsto \Psi_{K,p,q,s}(A, A)$ is convex for all unitary K and all n.

The same is true if convex is replaced by concave in all statements.