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The permanent-on-top conjecture is false

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ABSTRACT

A counterexample to the permanent-on-top conjecture for positive semidefinite Hermitian matrices is presented, which is a (complex) matrix of dimension $N = 5$ and rank $r = 2$.

1. Introduction

The permanent-on-top (POT) conjecture, formulated almost fifty years ago by Soules [1], states that the largest eigenvalue of the Schur power matrix of a positive semidefinite Hermitian matrix H is $\mathrm{per}(H)$, i.e., the matrix permanent of H (for a review see Refs. [2,3]). It is reminiscent of the famous result of Schur [4] stating that the smallest eigenvalue of the Schur power matrix is $\det(H)$. The POT conjecture is also

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a stronger version of the permanental-dominance conjecture of Lieb [5], formulated for characters of the symmetric group, which attracts a great deal of interest [7–11]. Bapat and Sunder [6] have proven the POT conjecture for matrices of dimension $N \leq 3$.

There is a quantum optics application of the POT conjecture: Important features of behavior of identical bosonic particles (e.g., photons) in a linear quantum network depend on the spectrum of the Schur power matrix of a positive semidefinite Hermitian matrix [12] (see also section 3).

Let us state the POT conjecture in some detail. Given a positive semidefinite Hermitian matrix H of dimension N, it states that $\mathrm{per}(H)$ is the largest eigenvalue of the matrix of diagonals of H, i.e., the positive semidefinite Hermitian [6] matrix of dimension $N!$ with elements indexed by permutations $\sigma, \tau \in \mathcal{S}_N$:

$$\Pi_{\sigma,\tau}(H) \equiv \prod_{k=1}^{N} H_{\sigma(k),\tau(k)}, \tag{1}$$

where $\mathcal{S}_N$ is the permutation (symmetric) group of N objects. The matrix in Eq. (1) is called the Schur power matrix.

It seems that a numerical test of the POT conjecture has never been attempted. Such numerical simulations are reported in the present work.

2. A counterexample to the POT

Counterexamples to the POT were sought for by generating a set of eigenvalues of H uniformly distributed in some finite interval ($[0, 1]$ was used), a Haar random unitary matrix of the eigenvectors of H, by the algorithm of Ref. [13], and analysing the eigenvalues of the Schur power matrix $\Pi(H)$. To have a non-zero probability of a rank deficient H, some eigenvalues were set to zero (leaving at least two non-zero, since it is easy to establish that for rank-1 H the conjecture is true). A counterexample matrix consisting of elements with integer real and imaginary parts was found in this way by first scaling up a numerical counterexample (more precisely, the unnormalized eigenvectors of H, see below) with a large number and rounding up the result to integers. A similar scaling operation applied to the eigenvector of the largest eigenvalue of $\Pi(H)$ allows to find a $N!$-dimensional vector X satisfying an inequality falsifying the POT,

$$X^\dagger \Pi(H) X > \mathrm{per}(H) X^\dagger X, \tag{2}$$

whose components have integer real and imaginary parts. Such a counterexample can be easily verified.